PERGAMON

International Journal of Solids and Structures 40 (2003) 3737–3785

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

Behavior of beams under transverse impact according to higher-order beam theory

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Received 30 October 2001; received in revised form 20 February 2002

Abstract

Using Reissner's principle, we formulated an equation of motion for a beam according to higher-order beam theory. We derived the Laplace transform of the equation and investigated wave-propagation behavior under transverse impact. In other words, we studied the effect of the nonlinear component of axial-warping, which cannot be determined by a conventional approach such as the Timoshenko beam theory. Specifically, we derived the transfer matrices for finite and semi-infinite beams. By choosing the appropriate state quantities, arrangement as a vector, and definition of sign convention, we were able to derive a perfect "reciprocal relation." In spite of the complicated Laplace inverse transform, we obtained an accurate and rapid solution by investigating appropriate branch points and poles and setting branch cuts. The extent of the nonlinear warping effect and its region of influence were also clearly demonstrated.

Keywords: Transverse impact; Higher-order beam theory; Laplace transform; Reciprocal relation; Nonlinear warping effect; Bromwich integral

1. Introduction

The analysis of beams under transverse impact has a long history in beam theory. The main methods employed in the analysis are mode superposition (Anderson, 1953), Laplace transform (Boley and Chao, 1955), characteristics-based analysis (Plass, 1958), wavefront expansion (Thambiratnam, 1984) and the finite element method (Yokoyama and Kishida, 1982). Among these, the finite element method (FEM) is well known as an efficient means of analyzing wave propagations in beams and/or structures with complicated boundaries. However, a large number of finite elements must be used to adequately model beams and structures. Furthermore, the higher the frequency, the larger the number of elements that must be used.

The Laplace transform technique yields a theoretically exact solution within the bounds established by the simplifying assumptions for the beam. However, because deriving the Laplace inverse transform is

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0020-7683/03/$ - see front matter
doi:10.1016/S0020-7683(03)00142-2

generally complicated and difficult, the Laplace transform technique is usually not the method of choice. In this study, we expanded the governing equation for the Timoshenko beam, which is generally thought to be a standard, into the governing equation for higher-order beam theory. Specifically, we formulated the equation of motion for any higher-order warping (Usuki and Maki, 2001) by applying the energy principle, which uses Reissner's functional. This provides greater accuracy when describing impact behavior. However, if we increase the approximation of the beam theory by only a single step, the formula becomes extremely complicated. In this study, we dealt with an equation of motion for a beam up to the first step as an example calculation; namely, we handled simultaneous partial differential equations having three unknown functions. Thus, the effect of nonlinear axial-warping could be taken into account.

For the Browmwich integral of a function that has been inversely transformed, key elements are to set up accurate branch points and poles on a complex plane and to introduce appropriate branch-cutting. The contribution from the integral values can be given by branch cuts, by branch points, and/or by poles. Thus, we studied the mechanism of how the integral value is given in each case. In analysis of the beam-impact problem, fundamental solutions to a governing differential equation for the beam play an important role. As a function of complex variable p of Laplace transform, the roots of the characteristic equation vary along the branch cut, with roles changing at the branch points. For this integral calculation, we pursued various approaches in an attempt to derive a simple, correct theoretical analysis, and we succeeded in achieving exact results. By comparing our results with the conventional Timoshenko beam calculation as well as with the numerical integral, we were able to verify the exactness of our calculation.

In the ordinary transfer matrix method in time domain, if we properly define the state quantities and their sign convention, we can formulate a "reciprocal relation" for which the transfer matrix element becomes symmetrical with respect to the subsidiary diagonal line. In this study, we showed that this relation can exist in the Laplace-transformed space. In addition, we numerically proved that this "reciprocal relation" can be established in the inversely transformed space as well. Moreover, we confirmed this from the distribution of a semi-infinite beam state function in the axial direction, as an example numerical calculation. Then, we numerically calculated the extent of the effect of the nonlinear warping in the axial direction and the range of the effect.

2. Coordinate system

We chose a coordinate system in which the x axis passes through the neutral axis of a beam of constant height h, width b, and axial length L; the y axis, which is horizontal in the coordinate space, perpendicularly intersects the x axis; and the z axis is in the beam-height direction. These x, y, and z axes form a right-handed coordinate system. The z axis is perpendicularly downward of the y axis in a clockwise direction. The displacement components for the x, y, z directions at point $P(x, y, z)$ in the beam are denoted by u, v, w, respectively (Fig. 1).

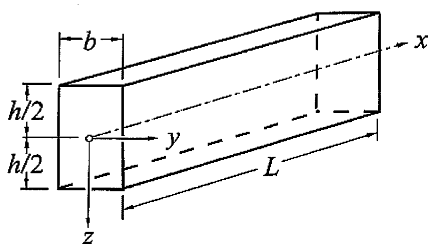

Fig. 1. Coordinate system.

The theory developed here is a restricted version for bending about a principal axis y, derived from Usuki's general theory for thin-walled beams (Usuki and Sawada, 1999) and thin plates (Usuki and Maki, 2000). To study the fundamentals of beam impact behavior, a simple rectangular cross section is treated in this paper. As the theory can be used to estimate arbitrarily higher-order shear lag, the applicable range of beam ratio h/L is greater than 5. In order to avoid cross-sectional distortion, the applicable range of beam ratio b/h is limited to less than 2.

3. Governing equation

3.1. Basic conditions

According to the infinitesimal displacement theory, the relation between strain and displacement becomes (Timoshenko and Goodier, 1970):

$$\varepsilon_{xx} = \frac{\partial u}{\partial x}, \quad \varepsilon_{zz} = \frac{\partial w}{\partial z}, \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}. \tag{1}$$

Constitutive equations with Young's modulus E, shear modulus G and Poisson's ratio v are:

$$\sigma_{xx} = E\varepsilon_{xx}, \quad \sigma_{zz} = E\varepsilon_{zz}, \quad \tau_{zx} = G\gamma_{zx}. \tag{2}$$

Following the engineering bending theory of beams, we ignored the components of transversal strain components with Poisson's ratio v.

Denoting the i direction component of the body force by ρb_i, the equilibrium equation for stress in an infinitesimal rectangular parallelepiped is

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{zx}}{\partial z} + \rho b_x = \rho \frac{\partial^2 u}{\partial t^2}, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} + \rho b_z = \rho \frac{\partial^2 w}{\partial t^2}, \end{aligned}\right\} \tag{3}$$

where ρ denotes the mass density of the body and t denotes time.

3.2. Equation of motion for a beam

The shear stress τ_{zx} is obtained from the equilibrium equation (3a) of stress by integration in the z direction. From this result, the shear strain γ_{zx} is given by the constitutive equation (2c). Therefore, although this shear stress satisfies the equilibrium equations, it does not satisfy the strain–displacement relation (1c). The axial warping function $u(t,x,z)$ is corrected so as to satisfy the strain–displacement relation (1c) for a fixed cross-sectional displacement function $w(t,x,z)$. From this corrected warping function, the axial strain and axial stress of first-step correction can be obtained. After the Nth step correction of this operation, displacement functions are obtained as follows (Usuki and Sawada, 1999; Usuki and Maki, 2000).

When the vector of the unit-warping function in the axial direction of the beam is expressed as $\boldsymbol{Z}(z)$ and the vector of the cross-sectional rotation accompanying bending deformation of the beam is expressed as $\boldsymbol{\theta}(t,x)$, the displacement is determined as

$$\left.\begin{aligned} u(t,x,z) &= -\boldsymbol{Z}^{\mathrm{T}}(z)\boldsymbol{\theta}(t,x), \\ w(t,x,z) &= w(t,x). \end{aligned}\right\} \tag{4}$$

Here, displacement in the beam-height direction, w, is independent of z; i.e., the coordinate along the height axis. Therefore, this is substituted into the strain displacement relation given by (1), yielding:

$$\left.\begin{aligned}\varepsilon_{xx}(t,x,z) &= -\boldsymbol{Z}^{\mathrm{T}}(z)\frac{\partial}{\partial x}\boldsymbol{\theta}(t,x),\\ \gamma_{xz}(t,x,z) &= \frac{\partial w(t,x)}{\partial x} - \frac{\partial}{\partial z}\boldsymbol{Z}^{\mathrm{T}}(z)\boldsymbol{\theta}(t,x),\\ \varepsilon_{zz}(t,x,z) &= 0.\end{aligned}\right\}\tag{5}$$

Components of the vector of the unit warping function, $\boldsymbol{Z}(z)$, the vector of the unit shearing function, $\boldsymbol{S}(z)$, and the vector of cross-sectional rotation $\boldsymbol{\theta}(t,x)$ of the beam are given as

$$\left.\begin{aligned}\boldsymbol{Z}^{\mathrm{T}}(z) &= [\,Z_0(z) \quad Z_1(z) \quad \cdots \quad Z_N(z)\,],\\ \boldsymbol{S}^{\mathrm{T}}(z) &= [\,S_0(z) \quad S_1(z) \quad \cdots \quad S_N(z)\,],\\ \boldsymbol{\theta}^{\mathrm{T}}(t,x) &= [\,\theta_{y0}(t,x) \quad \theta_{y1}(t,x) \quad \cdots \quad \theta_{yN}(t,x)\,].\end{aligned}\right\}\tag{6}$$

The unit warping function at the zeroth step is the same as that in engineering beam theory; namely

$$Z_0(z) = z \tag{7}$$

and the corresponding cross-sectional rotation is $\theta_{y0}(t,x)$. Elements of the unit warping function vector, $\boldsymbol{Z}(z)$, beyond the zeroth step are set to be mutually orthogonal (Usuki and Sawada, 1999).

By substituting the longitudinal displacement equation (4a) into Hooke's law, given by Eq. (2), we obtain the normal stress $\sigma_{xx}(t,x,z)$ in the axial direction. Then, by substituting the result into the equilibrium condition of stress in the axial direction, given by Eq. (3a), we obtain the shearing stress, $\tau_{zx}(t,x,z)$. When no inertial force or body force is present, the equation becomes

$$\left.\begin{aligned}\sigma_{xx}(t,x,z) &= -E\boldsymbol{Z}^{\mathrm{T}}(z)\frac{\partial}{\partial x}\boldsymbol{\theta}(t,x),\\ \tau_{xz}(t,x,z) &= E\frac{\boldsymbol{S}^{\mathrm{T}}(z)}{b(z)}\frac{\partial^2}{\partial x^2}\boldsymbol{\theta}(t,x),\end{aligned}\right\}\tag{8}$$

where the unit shearing function is set to

$$\boldsymbol{S}(z) = \int_{-h/2}^{z}\boldsymbol{Z}(z)\,\mathrm{d}A. \tag{9}$$

The warping moment vector $\boldsymbol{M}(t,x)$ and the shearing force vector $\boldsymbol{Q}(t,x)$ are defined as the integral over the sectional area of the beam cross section A, or as

$$\left.\begin{aligned}\boldsymbol{M}(t,x) &= \textstyle\int_A \sigma_{xx}(t,x)\boldsymbol{Z}(z)\,\mathrm{d}A,\\ \boldsymbol{Q}(t,x) &= \textstyle\int_A \tau_{xz}(t,x)\dot{\boldsymbol{Z}}(z)\,\mathrm{d}A.\end{aligned}\right\}\tag{10}$$

Components of these vectors are denoted as

$$\left.\begin{aligned}\boldsymbol{M}^{\mathrm{T}}(t,x) &= [\,M_{y0}(t,x) \quad M_{y1}(t,x) \quad \cdots \quad M_{yN}(t,x)\,],\\ \boldsymbol{Q}^{\mathrm{T}}(t,x) &= [\,Q_{z0}(t,x) \quad Q_{z1}(t,x) \quad \cdots \quad Q_{zN}(t,x)\,].\end{aligned}\right\}\tag{11}$$

By substituting Eq. (8) into the stress component in the right-hand side of Eq. (10), we express the stress resultants as

$$\left.\begin{aligned}\boldsymbol{M}(t,x) &= -E\boldsymbol{F}\frac{\partial}{\partial x}\boldsymbol{\theta}(t,x),\\ \boldsymbol{Q}(t,x) &= -E\boldsymbol{F}\frac{\partial^2}{\partial x^2}\boldsymbol{\theta}(t,x).\end{aligned}\right\}\tag{12}$$

In this equation, the warping resistance matrix $\boldsymbol{F}$ is defined as

$$\boldsymbol{F} = \int_A \boldsymbol{Z}(z)\boldsymbol{Z}^{\mathrm{T}}(z)\,\mathrm{d}A. \tag{13}$$

In matrix form, this is

$$\boldsymbol{F} = \begin{bmatrix} I_{y0} & & & \mathbf{0} \\ & I_{y1} & & \\ & & \ddots & \\ \mathbf{0} & & & I_{yN} \end{bmatrix}. \tag{14}$$

The first element I_{y0} of the diagonal elements is the moment of inertia of bending rotation around the y axis of the cross section. The unit warping functions beyond the zeroth step are normalized so that the total warping resistance is equal to the moment of inertia of the cross section.

The displacement components on the right-hand side of Eq. (12) are expressed using the stress resultants on the left-hand side. Then, when this is substituted into Eq. (8) of stress, the results are as follows:

$$\left.\begin{aligned} \sigma_{xx}(t,x,z) &= \boldsymbol{Z}^{\mathrm{T}}(z)\boldsymbol{F}^{-1}\boldsymbol{M}(t,x), \\ \tau_{xz}(t,x,z) &= -\frac{\boldsymbol{S}^{\mathrm{T}}(z)}{b(z)}\boldsymbol{F}^{-1}\boldsymbol{Q}(t,x). \end{aligned}\right\} \tag{15}$$

In order to derive an equation of motion for a beam, we applied Hamilton's principle in conjunction with Reissner's functional ψ (Vinson and Chou, 1975). Given that the kinetic energy of the structural system from the beginning of time t_0 until t_1 is T, the functional given below takes the extreme value when the boundary condition, the equilibrium condition of stress, and the stress displacement relation are satisfied.

$$\Phi = \int_{t_0}^{t_1} (T - \psi)\,\mathrm{d}t. \tag{16}$$

That is, the following holds:

$$\delta\Phi = 0. \tag{17}$$

The kinetic energy of a beam bending, T, is expressed as

$$T = \int_0^L \int_A \frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}A\,\mathrm{d}x. \tag{18}$$

When we substitute displacement equation (4) into the above equation and integrate with respect to the independent variable z in the beam-height direction, we obtain

$$T = \int_0^L \frac{1}{2}\rho\left[\frac{\partial\boldsymbol{\theta}^{\mathrm{T}}}{\partial t}\boldsymbol{F}\frac{\partial\boldsymbol{\theta}}{\partial t} + \frac{\partial w}{\partial t}A\frac{\partial w}{\partial t}\right]\mathrm{d}x. \tag{19}$$

Reissner's functional is shown as

$$\psi = \int_R H\,\mathrm{d}R - \int_R F_i u_i\,\mathrm{d}R - \int_{S_t} T_i u_i\,\mathrm{d}S, \tag{20}$$

where S_t is the portion of S on which stresses are prescribed; $H = \sigma_{ij}\varepsilon_{ij} - W(\sigma_{ij})$ and $W(\sigma_{ij})$ is the strain energy function in terms of stresses only.

The strain energy function in terms of stresses only is written as follows. For an isotropic material:

$$W = \frac{1}{2E}\left[\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - 2\nu(\sigma_{xx}\sigma_{yy} + \sigma_{yy}\sigma_{zz} + \sigma_{zz}\sigma_{xx}) + 2(1+\nu)\left(\sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2\right)\right]. \tag{21}$$

In the case of a beam subjected to a distributed transverse load $q(t,x)$, a concentrated transverse load Q_{z0}^*, a distributed axial load $p(x,z,t)$, and a concentrated axial load P_x, Reissner's functional is

$$\begin{aligned}\psi = \int_0^L \int_A \Bigg\{ &-\sigma_{xx}\boldsymbol{Z}^{\mathrm{T}}\frac{\partial\boldsymbol{\theta}}{\partial x} + \sigma_{xz}\left(\frac{\partial w}{\partial x} - \frac{\partial\boldsymbol{Z}^{\mathrm{T}}}{\partial z}\boldsymbol{\theta}\right) - \frac{1}{2E}\Bigg[\left(\boldsymbol{M}^{\mathrm{T}}\boldsymbol{F}^{-1}\boldsymbol{Z}\right)\left(\boldsymbol{Z}^{\mathrm{T}}\boldsymbol{F}^{-1}\boldsymbol{M}\right) \\ &+ 2(1+\nu)\left(\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{F}^{-1}\frac{\boldsymbol{S}}{b(z)}\right)\left(\frac{\boldsymbol{S}^{\mathrm{T}}}{b(z)}\boldsymbol{F}^{-1}\boldsymbol{Q}\right)\Bigg]\Bigg\}\mathrm{d}A\,\mathrm{d}x \\ &- \int_0^L \left(q(t,x)w - \int_A p(t,x)u\,\mathrm{d}A\right)\mathrm{d}x - \left(Q_{z0}^*w - P_x u\right).\end{aligned} \tag{22}$$

Here, we introduce a base vector $\boldsymbol{e}_1$ such that only the first element is 1 and the other elements are zero. Namely, the base vector is defined as

$$\boldsymbol{e}_1^{\mathrm{T}} \equiv [1 \quad 0 \quad \cdots \quad 0]. \tag{23}$$

By combining this base vector and the unit warping function vector equation (6a) and considering Eq. (7), we obtain the relation

$$\frac{\partial\boldsymbol{Z}^{\mathrm{T}}}{\partial z}\boldsymbol{e}_1 \equiv 1. \tag{24}$$

Premultiplying this, as a dummy, by $\partial w/\partial x$ of Eq. (22) and integrating over the area of the cross section, we obtain

$$\begin{aligned}\psi = \int_0^L \Bigg\{ &-\boldsymbol{M}^{\mathrm{T}}\frac{\partial\boldsymbol{\theta}}{\partial x} + \boldsymbol{Q}^{\mathrm{T}}\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right) - \left[\frac{1}{2}\boldsymbol{M}^{\mathrm{T}}(E\boldsymbol{F})^{-1}\boldsymbol{M} + \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}(GA\boldsymbol{k}')^{-1}\boldsymbol{Q}\right]\Bigg\}\mathrm{d}x \\ &- \int_0^L \left(q(t,x)w - \boldsymbol{m}^{\mathrm{T}}(t,x)\boldsymbol{\theta}\right)\mathrm{d}x - \left(Q_{z0}^*w - \boldsymbol{M}^{*\mathrm{T}}\boldsymbol{\theta}\right).\end{aligned} \tag{25}$$

The coefficient matrix of the shear correction $\boldsymbol{k}'$ in the above equation is the symmetry matrix, which is defined as

$$\boldsymbol{k}' = \frac{1}{A}\boldsymbol{F}\boldsymbol{R}^{-1}\boldsymbol{F}. \tag{26}$$

The shearing resistance matrix $\boldsymbol{R}$ on the right-hand side can be calculated from the equation

$$\boldsymbol{R} = \int_A \frac{\boldsymbol{S}(z)}{b(z)}\frac{\boldsymbol{S}^{\mathrm{T}}(z)}{b(z)}\,\mathrm{d}A. \tag{27}$$

Specifically, the matrix can be denoted as

$$\boldsymbol{R} = \begin{bmatrix} R_{00} & R_{01} & 0 & & & & \\ R_{10} & R_{11} & R_{12} & \ddots & & \mathbf{0} & \\ 0 & R_{21} & R_{22} & \ddots & & & \\ & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \\ & & & \ddots & R_{N-2,N-2} & R_{N-2,N-1} & 0 \\ & \mathbf{0} & & \ddots & R_{N-1,N-2} & R_{N-1,N-1} & R_{N-1,N} \\ & & & & 0 & R_{N,N-1} & R_{N,N} \end{bmatrix}. \tag{28}$$

This becomes a symmetric band matrix which has nonzero values only in the diagonal elements and in the elements to their immediate right and left sides.

The distributed bending couple $\boldsymbol{m}(t,x)$ and the concentrated bending couple $\boldsymbol{M}^*(t,x)$ are defined as

$$\left.\begin{aligned} \boldsymbol{m}(t,x) &= -\textstyle\int_A p(t,x,z)\boldsymbol{Z}(z)\,\mathrm{d}A, \\ \boldsymbol{M}^*(t,x) &= -P_x(t,x,z)\boldsymbol{Z}(z). \end{aligned}\right\} \tag{29}$$

By substituting the functional (25) and the kinetic energy equation (19) into Eq. (16), we obtain

$$\begin{aligned} \Phi = \int_{t_0}^{t_1}\int_0^L \Bigg\{ \frac{1}{2}\rho\left[\frac{\partial\boldsymbol{\theta}^{\mathrm{T}}}{\partial t}\boldsymbol{F}\frac{\partial\boldsymbol{\theta}}{\partial t} + \frac{\partial w}{\partial t}A\frac{\partial w}{\partial t}\right] + \boldsymbol{M}^{\mathrm{T}}\boldsymbol{\theta}' - \boldsymbol{Q}^{\mathrm{T}}\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right) \\ + \left[\frac{1}{2}\boldsymbol{M}^{\mathrm{T}}(E\boldsymbol{F})^{-1}\boldsymbol{M} + \frac{1}{2}\boldsymbol{Q}^{\mathrm{T}}(GA\boldsymbol{k}')^{-1}\boldsymbol{Q} - q(t,x)w + \boldsymbol{m}^{\mathrm{T}}(t,x)\boldsymbol{\theta}\right]\Bigg\}\mathrm{d}x\,\mathrm{d}t - Q_{z0}^*w + \boldsymbol{M}^{*\mathrm{T}}\boldsymbol{\theta}. \end{aligned} \tag{30}$$

Taking the variation under condition (17) that Eq. (30) takes the extreme value, we obtain the motion equation and the relation between the stress resultants and displacements as follows

$$\frac{\partial\boldsymbol{M}}{\partial x} - \boldsymbol{Q} - \boldsymbol{m}(t,x) + \rho\boldsymbol{F}\frac{\partial^2\boldsymbol{\theta}}{\partial t^2} = \mathbf{0}, \tag{31}$$

$$\frac{\partial Q_{z0}}{\partial x} + q(t,x) - \rho A\frac{\partial^2 w}{\partial t^2} = 0, \tag{32}$$

$$(E\boldsymbol{F})^{-1}\boldsymbol{M} + \frac{\partial\boldsymbol{\theta}}{\partial x} = \mathbf{0}, \tag{33}$$

$$(GA\boldsymbol{k}')^{-1}\boldsymbol{Q} - \left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right) = \mathbf{0}. \tag{34}$$

Eqs. (33) and (34) can also be written as the constitutive equations of stress resultants.

$$\boldsymbol{M} = -E\boldsymbol{F}\frac{\partial\boldsymbol{\theta}}{\partial x}, \tag{35}$$

$$\boldsymbol{Q} = GA\boldsymbol{k}'\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right). \tag{36}$$

The relation (12b) between shearing force $\boldsymbol{Q}$ and the rotation of cross section $\boldsymbol{\theta}$ is transformed to the constitutive equation (36) of a Timoshenko type beam. As shown in formula (11b), the shearing force Q_{z0} of the zeroth step is the first element of the vector of shearing force $\boldsymbol{Q}$. Thus, we eliminate the first element from the right-hand side of formula (36) by premultipling the vector $\boldsymbol{e}_1^{\mathrm{T}}$ as shown in the equation below.

$$Q_{z0} = GA\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{k}'\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right). \tag{37}$$

The boundary condition is as follows.

$$\left.\begin{array}{l}[\delta\boldsymbol{\theta}(\boldsymbol{M} - \boldsymbol{M}^*)]_0^L = \boldsymbol{0}, \\ \left[\delta w\left(Q_{z0} - Q_{z0}^*\right)\right]_0^L = 0.\end{array}\right\} \tag{38}$$

3.3. Differential equation with one deformation vector

By substituting the constitutive equations (35)–(37) into the governing equations of motion (31) and (32) for stress resultants and the displacement components, we obtain the governing equations for the displacement components as

$$\frac{\partial}{\partial x}\left[E\boldsymbol{F}\frac{\partial\boldsymbol{\theta}}{\partial x}\right] + GA\boldsymbol{k}'\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right) - \rho\boldsymbol{F}\frac{\partial^2\boldsymbol{\theta}}{\partial t^2} = -\boldsymbol{m}(t,x), \tag{39}$$

$$\frac{\partial}{\partial x}\left[GA\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{k}'\left(\boldsymbol{e}_1\frac{\partial w}{\partial x} - \boldsymbol{\theta}\right)\right] - \rho A\frac{\partial^2 w}{\partial t^2} = -q(t,x). \tag{40}$$

In the case of a constant cross section along the direction of the beam axis, if we delete the deflection w from these two simultaneous differential equations, we obtain the formula of the rotation vector $\boldsymbol{\theta}$:

$$\begin{aligned}&\left\{\left(E\boldsymbol{F}\frac{\partial^2}{\partial x^2} - GA\boldsymbol{k}' - \rho\boldsymbol{F}\frac{\partial^2}{\partial t^2}\right)\left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{Gk}\frac{\partial^2}{\partial t^2}\right) + GA\frac{\boldsymbol{k}'}{k}\boldsymbol{e}_1\boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{k}'\frac{\partial^2}{\partial x^2}\right\}\boldsymbol{\theta} \\ &\quad = \frac{\boldsymbol{k}'}{k}\boldsymbol{e}_1\frac{\partial}{\partial x}q(t,x) - \left(\frac{\partial^2}{\partial x^2} - \frac{\rho}{Gk}\frac{\partial^2}{\partial t^2}\right)\boldsymbol{m}(t,x),\end{aligned} \tag{41}$$

k in the above formula indicates element k_{00} in the first column of the first row within the coefficient matrix of shear correction $\boldsymbol{k}'$. The element of this matrix can be extracted by multiplying from the front and back of the coefficient matrix by the base vector $\boldsymbol{e}_1$; namely,

$$k \equiv k_{00} = \boldsymbol{e}_1^{\mathrm{T}}\boldsymbol{k}'\boldsymbol{e}_1. \tag{42}$$

Warping resistance $\boldsymbol{F}$ is a diagonal matrix; however, the coefficient matrix of shear correction $\boldsymbol{k}'$ is generally a symmetric matrix in which all elements have nonzero values. Therefore, in order to formulate a differential equation for a single deformation quantity, we have to transform it by elimination. That is, we start with a Bernoulli/Euler beam and a Timoshenko beam of fourth-order; the number of orders of the differential equation for a single deformation quantity will increase by two orders every time we consider a shear-lag of one step.

From here on, we explain the theory and give a numerical example, considering up to the first step of shear-lag. If we can accept the complexity of calculation and are willing to go further, we can use the same method to obtain a solution that considers the second step of shear-lag or beyond.

3.4. Differential equation of one deformation quantity

When we consider only up to the first step of shear-lag, the main vectors are as follows:

$$\left.\begin{aligned} \boldsymbol{M}(t,x) &= \begin{bmatrix} M_{y0}(t,x) \\ M_{y1}(t,x) \end{bmatrix}, \\ \boldsymbol{Q}(t,x) &= \begin{bmatrix} Q_{z0}(t,x) \\ Q_{z1}(t,x) \end{bmatrix}, \\ \boldsymbol{\theta}(t,x) &= \begin{bmatrix} \theta_{y0}(t,x) \\ \theta_{y1}(t,x) \end{bmatrix}, \\ \boldsymbol{F} &= \begin{bmatrix} I_{y0} & 0 \\ 0 & I_{y1} \end{bmatrix}, \\ \boldsymbol{k}' &= \begin{bmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{bmatrix}, \\ \boldsymbol{Z}(z) &= \begin{bmatrix} Z_0(z) \\ Z_1(z) \end{bmatrix}, \\ \boldsymbol{e}_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned}\right\} \tag{43}$$

The coefficient matrix of shear correction $\boldsymbol{k}'$ is symmetric; thus, $k_{01} = k_{10}$. In addition, the diagonal element of the warping resistance matrix $\boldsymbol{F}$ is

$$I_{y1} = I_{y0}. \tag{44}$$

Thus, we normalize the unit warping function $Z_1(z)$.

Hereafter, the warping moment $\boldsymbol{M}(t,x)$ and the shearing force $\boldsymbol{Q}(t,x)$ can be expressed as

$$\left.\begin{aligned} \begin{bmatrix} M_{y0}(t,x) \\ M_{y1}(t,x) \end{bmatrix} &= -E\begin{bmatrix} I_{y0} & 0 \\ 0 & I_{y1} \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \theta_{y0} \\ \theta_{y1} \end{bmatrix}, \\ \begin{bmatrix} Q_{z0}(t,x) \\ Q_{z1}(t,x) \end{bmatrix} &= GA\begin{bmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial w}{\partial x} - \theta_{y0} \\ -\theta_{y1} \end{bmatrix}. \end{aligned}\right\} \tag{45}$$

Governing Eqs. (39) and (40) can be presented as

$$\left.\begin{aligned} &EI_{y0}\frac{\partial^2\theta_{y0}}{\partial x^2} + GAk_{00}\left(\frac{\partial w}{\partial x} - \theta_{y0}\right) + GAk_{01}(-\theta_{y1}) - \rho I_{y0}\frac{\partial^2\theta_{y0}}{\partial t^2} = -m_{y0}, \\ &EI_{y1}\frac{\partial^2\theta_{y1}}{\partial x^2} + GAk_{10}\left(\frac{\partial w}{\partial x} - \theta_{y0}\right) + GAk_{11}(-\theta_{y1}) - \rho I_{y1}\frac{\partial^2\theta_{y1}}{\partial t^2} = -m_{y1}, \\ &GAk_{00}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial\theta_{y0}}{\partial x}\right) + GAk_{01}\left(-\frac{\partial\theta_{y1}}{\partial x}\right) - \rho A\frac{\partial^2 w}{\partial t^2} = -q. \end{aligned}\right\} \tag{46}$$

In order to make this formula dimensionless, we define the following quantities.

$$x_1 = \frac{x}{r}, \quad w_1 = \frac{w}{r}, \quad t_1 = \frac{c_1}{r}t, \quad \gamma = \frac{E}{Gk}, \quad r = \left(\frac{I_{y0}}{A}\right)^{1/2}, \quad c_1 = \left(\frac{E}{\rho}\right)^{1/2}, \quad c_2 = \left(\frac{Gk}{\rho}\right)^{1/2}. \tag{47}$$

Using these, we can rewrite the formula of stress resultants (45)

$$\left.\begin{aligned} r\begin{bmatrix} M_{y0}(t_1,x_1) \\ M_{y1}(t_1,x_1) \end{bmatrix} &= -E\begin{bmatrix} I_{y0} & 0 \\ 0 & I_{y1} \end{bmatrix} \frac{\partial}{\partial x_1} \begin{bmatrix} \theta_{y0} \\ \theta_{y1} \end{bmatrix}, \\ \begin{bmatrix} Q_{z0}(t_1,x_1) \\ Q_{z1}(t_1,x_1) \end{bmatrix} &= GA\begin{bmatrix} k_{00} & k_{01} \\ k_{10} & k_{11} \end{bmatrix} \begin{bmatrix} \frac{\partial w_1}{\partial x_1} - \theta_{y0} \\ -\theta_{y1} \end{bmatrix}. \end{aligned}\right\} \tag{48}$$

In addition, we define dimensionless stress resultants $\overline{M}_{y0}$, $\overline{M}_{y1}$, $\overline{Q}_{z0}$, and $\overline{Q}_{z1}$.

$$\left.\begin{aligned}\begin{bmatrix}\overline{M}_{y0}(t_1,x_1)\\ \overline{M}_{y1}(t_1,x_1)\end{bmatrix} &\equiv \frac{r}{EI_{y0}}\begin{bmatrix}M_{y0}(t_1,x_1)\\ M_{y1}(t_1,x_1)\end{bmatrix} = -\frac{\partial}{\partial x_1}\begin{bmatrix}\theta_{y0}\\ \theta_{y1}\end{bmatrix},\\ \begin{bmatrix}\overline{Q}_{z0}(t_1,x_1)\\ \overline{Q}_{z1}(t_1,x_1)\end{bmatrix} &\equiv \frac{\gamma^{-1}}{GAk}\begin{bmatrix}Q_{z0}(t_1,x_1)\\ Q_{z1}(t_1,x_1)\end{bmatrix} = \frac{\gamma^{-1}}{k}\begin{bmatrix}k_{00} & k_{01}\\ k_{10} & k_{11}\end{bmatrix}\begin{bmatrix}\dfrac{\partial w_1}{\partial x_1}-\theta_{y0}\\ -\theta_{y1}\end{bmatrix}.\end{aligned}\right\} \tag{49}$$

The governing equations for deformation quantities (46) can also be rewritten using Eq. (47).

$$\left.\begin{aligned}&\frac{\partial^2\theta_{y0}}{\partial x_1^2}+\frac{1}{\gamma}\left(\frac{\partial w_1}{\partial x_1}-\theta_{y0}\right)+\frac{1}{\gamma}\frac{k_{01}}{k_{00}}(-\theta_{y1})-\frac{\partial^2\theta_{y0}}{\partial t_1^2}=-\frac{m_{y0}}{EA},\\ &\frac{\partial^2\theta_{y1}}{\partial x_1^2}+\frac{1}{\gamma}\frac{k_{10}}{k_{00}}\left(\frac{\partial w_1}{\partial x_1}-\theta_{y0}\right)+\frac{1}{\gamma}\frac{k_{11}}{k_{00}}(-\theta_{y1})-\frac{\partial^2\theta_{y1}}{\partial t_1^2}=-\frac{m_{y1}}{EA},\\ &\frac{1}{\gamma}\left(\frac{\partial^2 w_1}{\partial x_1^2}-\frac{\partial\theta_{y0}}{\partial x_1}\right)+\frac{1}{\gamma}\frac{k_{01}}{k_{00}}\left(-\frac{\partial\theta_{y1}}{\partial x_1}\right)-\frac{\partial^2 w_1}{\partial t_1^2}=-q(t_1,x_1).\end{aligned}\right\} \tag{50}$$

We will use the Laplace transform (Doetsch, 1970). That is, if we define the following equation,

$$\left.\begin{aligned}W(p,x_1) &= \int_0^\infty w_1(t_1,x_1)\mathrm{e}^{-pt_1}\,\mathrm{d}t_1,\\ \begin{bmatrix}\Theta_{y0}(p,x_1)\\ \Theta_{y1}(p,x_1)\end{bmatrix} &= \int_0^\infty \begin{bmatrix}\theta_{y0}(t_1,x_1)\\ \theta_{y1}(t_1,x_1)\end{bmatrix}\mathrm{e}^{-pt_1}\,\mathrm{d}t_1,\\ \begin{bmatrix}\mathscr{M}_{y0}(p,x_1)\\ \mathscr{M}_{y1}(p,x_1)\end{bmatrix} &= \int_0^\infty \begin{bmatrix}\overline{M}_{y0}(t_1,x_1)\\ \overline{M}_{y1}(t_1,x_1)\end{bmatrix}\mathrm{e}^{-pt_1}\,\mathrm{d}t_1,\\ \begin{bmatrix}\mathscr{Q}_{z0}(p,x_1)\\ \mathscr{Q}_{z1}(p,x_1)\end{bmatrix} &= \int_0^\infty \begin{bmatrix}\overline{Q}_{z0}(t_1,x_1)\\ \overline{Q}_{z1}(t_1,x_1)\end{bmatrix}\mathrm{e}^{-pt_1}\,\mathrm{d}t_1,\end{aligned}\right\} \tag{51}$$

the derived functions for dimensionless time t_1 of deformation quantities are changed to

$$\left.\begin{aligned}pW(p,x_1) &= \int_0^\infty \frac{\partial}{\partial t_1}w_1(t_1,x_1)\mathrm{e}^{-pt_1}\,\mathrm{d}t_1,\\ p\begin{bmatrix}\Theta_{y0}(p,x_1)\\ \Theta_{y1}(p,x_1)\end{bmatrix} &= \int_0^\infty \frac{\partial}{\partial t_1}\begin{bmatrix}\theta_{y0}(t_1,x_1)\\ \theta_{y1}(t_1,x_1)\end{bmatrix}\mathrm{e}^{-pt_1}\,\mathrm{d}t_1.\end{aligned}\right\} \tag{52}$$

Then, the relation between the stress resultants and the deformation quantities after Laplace transformation can be expressed as

$$\left.\begin{aligned}\begin{bmatrix}\mathscr{M}_{y0}(p,x_1)\\ \mathscr{M}_{y1}(p,x_1)\end{bmatrix} &= -\frac{\partial}{\partial x_1}\begin{bmatrix}\Theta_{y0}(p,x_1)\\ \Theta_{y1}(p,x_1)\end{bmatrix},\\ \begin{bmatrix}\mathscr{Q}_{z0}(p,x_1)\\ \mathscr{Q}_{z1}(p,x_1)\end{bmatrix} &= \frac{\gamma^{-1}}{k}\begin{bmatrix}k_{00} & k_{01}\\ k_{10} & k_{11}\end{bmatrix}\begin{bmatrix}\dfrac{\partial W}{\partial x_1}-\Theta_{y0}(p,x_1)\\ -\Theta_{y1}(p,x_1)\end{bmatrix}.\end{aligned}\right\} \tag{53}$$

According to formula (50), the equation of motion, which was represented by the deformation quantities after transformation, will be as follows:

$$\left.\begin{aligned}&\frac{\partial^2\Theta_{y0}}{\partial x_1^2}+\frac{1}{\gamma}\left(\frac{\partial W}{\partial x_1}-\Theta_{y0}\right)+\frac{1}{\gamma}\frac{k_{01}}{k_{00}}(-\Theta_{y1})-p^2\Theta_{y0}=0,\\ &\frac{\partial^2\Theta_{y1}}{\partial x_1^2}+\frac{1}{\gamma}\frac{k_{10}}{k_{00}}\left(\frac{\partial W}{\partial x_1}-\Theta_{y0}\right)+\frac{1}{\gamma}\frac{k_{11}}{k_{00}}(-\Theta_{y1})-p^2\Theta_{y1}=0,\\ &\frac{1}{\gamma}\left(\frac{\partial^2 W}{\partial x_1^2}-\frac{\partial\Theta_{y0}}{\partial x_1}\right)+\frac{1}{\gamma}\frac{k_{01}}{k_{00}}\left(-\frac{\partial\Theta_{y1}}{\partial x_1}\right)-p^2 W=0.\end{aligned}\right\} \tag{54}$$

The distributed loads $q(t,x)$, $m_{y0}(t,x)$ and $m_{y1}(t,x)$ are now set to zero. By transposing formula (54a), we express deformation Θ_{y1} in terms of deformations W and Θ_{y0}.

$$\frac{1}{\gamma}\frac{k_{01}}{k_{00}}\Theta_{y1} = \frac{1}{\gamma}\frac{\partial W}{\partial x_1} + \left[\frac{\partial^2}{\partial x_1^2} - \left(\frac{1}{\gamma} + p^2\right)\right]\Theta_{y0}. \tag{55}$$

Substituting this into formula (54c), we can express deflection W in terms of Θ_{y0}.

$$W = \left(-\frac{1}{p^2}\frac{\partial^3}{\partial x_1^3} + \frac{\partial}{\partial x_1}\right)\Theta_{y0}. \tag{56}$$

By substituting this into formula (54b), we obtain the differential equation for deformation Θ_{y0} alone. We can also obtain in the same manner, differential equation for deformation Θ_{y1} alone or for deflection W alone, as shown below.

$$\left(\frac{\partial^6}{\partial x_1^6} + a\frac{\partial^4}{\partial x_1^4} + b\frac{\partial^2}{\partial x_1^2} + c\right)\begin{bmatrix} W(p,x_1) \\ \Theta_{y0}(p,x_1) \\ \Theta_{y1}(p,x_1)\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0\end{bmatrix}. \tag{57}$$

When we consider the first step of the shear-lag, the fourth-order differential equation of Timoshenko beam deflection w alone and the fourth-order differential equation of the rotation of cross section θ alone, each being equivalent to the zeroth step in consideration of shear-lag, become sixth-order differential equations. The coefficients for the differential equation are shown below.

$$\left.\begin{aligned} a &= -\left[\frac{1}{\gamma}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right) + p^2(2+\gamma)\right], \\ b &= p^2\left[\frac{1}{\gamma}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right) + \left(1 + \frac{k_{11}}{k_{00}}\right) + p^2(1+2\gamma)\right], \\ c &= -p^2\left[\frac{1}{\gamma}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right) + p^2\left(1 + \frac{k_{11}}{k_{00}}\right) + p^4\gamma\right]. \end{aligned}\right\} \tag{58}$$

As with the Timoshenko beam theory, the coefficients of the differential equation, a, b, and c, are common in this case regardless of the unknown deformation functions. This means that fundamental solutions for differential equations are common.

4. General solution of the governing equation

4.1. General solution of deformation

The characteristic equation of differential equation (57) is a sixth-order equation.

$$\lambda^6 + a\lambda^4 + b\lambda^2 + c = 0. \tag{59}$$

Or, if we let

$$\lambda^2 \equiv \Lambda, \tag{60}$$

the characteristic equation becomes a cubic equation, as shown below.

$$\Lambda^3 + a\Lambda^2 + b\Lambda + c = 0. \tag{61}$$

We solve this by Cardan's method (Bronshtein and Semendyayev, 1964). First, letting

$$\Lambda = \Lambda^* - \frac{a}{3}. \tag{62}$$

Eq. (61) is converted into a form with missing squared terms.

$$\Lambda^{*3} + 3p^*\Lambda^* + q^* = 0, \tag{63}$$

where,

$$\left.\begin{aligned} p^* &= \frac{1}{3}\left(b - \frac{a^2}{3}\right), \\ q^* &= c - \frac{1}{3}ab + \frac{2}{27}a^3. \end{aligned}\right\} \tag{64}$$

In order to express the new coefficients, p^* and q^*, with shear correction factors, we substitute formula (58) into the right side of the above formula.

$$p^* = -\frac{1}{9\gamma^2}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)^2 + \frac{p^2}{3}\left[\left(1 + \frac{k_{11}}{k_{00}}\right) - \frac{1}{3\gamma}(1+2\gamma)\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)\right] - \frac{p^4}{9}(-1+\gamma)^2, \tag{65}$$

$$\begin{aligned} q^* = &-\frac{2}{27\gamma^3}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)^3 + \frac{p^2}{3}\left[\frac{1}{\gamma}\left(-2 + \frac{k_{11}}{k_{00}}\right)\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right) - \frac{1}{3\gamma^2}(1+2\gamma)\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)^2\right] \\ &+ p^4\left[\frac{1}{3}(-1+\gamma)\left(1 + \frac{k_{11}}{k_{00}}\right) - \frac{1}{9\gamma}(-1+\gamma)(1+2\gamma)\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)\right] - p^6\left[\frac{2}{27}(-1+\gamma)^3\right]. \end{aligned} \tag{66}$$

Then, three roots of the cubic equation (63), which are the missing squared terms, are given as

$$\Lambda^* = -A - B, \quad -A\omega - B\omega^2, \quad -A\omega^2 - B\omega, \tag{67}$$

where

$$\left.\begin{aligned} A &= \left[\frac{1}{2}\left(q^* + \sqrt{q^{*2} + 4p^{*3}}\right)\right]^{1/3}, \\ B &= \left[\frac{1}{2}\left(q^* - \sqrt{q^{*2} + 4p^{*3}}\right)\right]^{1/3}, \\ AB &= -p^*, \end{aligned}\right\} \tag{68}$$

and ω is one of the imaginary roots in equation $x^3 = 1$; namely,

$$\omega = \tfrac{1}{2}(-1 + \mathrm{i}\sqrt{3}) \quad \text{or} \quad \tfrac{1}{2}(-1 - \mathrm{i}\sqrt{3}). \tag{69}$$

The square of either value of ω will be another value of ω. According to formula (62), the three roots of Λ of the first cubic equation are given as

$$\Lambda = -A - B - \frac{a}{3}, \quad -A\omega - B\omega^2 - \frac{a}{3}, \quad -A\omega^2 - B\omega - \frac{a}{3}. \tag{70}$$

From the sign of the value $q^{*2} + 4p^{*3}$ in the square root of the above Eq. (68), the real root or the complex root can be determined. Therefore, by providing symbol D as a discriminant, we express

$$D \equiv q^{*2} + 4p^{*3} = -\frac{d'}{27}p^2(p^6 + a'p^4 + b'p^2 + c') = -\frac{d'}{27}p^2(p^2 - p_1^2)(p^2 - p_2^2)(p^2 - p_3^2), \tag{71}$$

where the coefficients in the second equation of (71) are given as

$$
\left.
\begin{aligned}
a' &= -\frac{1}{d'}\left[4\left(1+\frac{k_{11}}{k_{00}}\right)^3 - \frac{2}{\gamma}\left(-4+10\gamma+(5+\gamma)\frac{k_{11}}{k_{00}}\right)\left(1+\frac{k_{11}}{k_{00}}\right)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)\right. \\
&\quad \left. +\frac{2}{\gamma^2}\left(1-\gamma+6\gamma^2+(4+2\gamma)\frac{k_{11}}{k_{00}}\right)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^2 - \frac{2}{\gamma^3}(1+\gamma)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^3\right], \\
b' &= \frac{1}{d'}\left[\frac{1}{\gamma^2}\left(-8+20\frac{k_{11}}{k_{00}}+\frac{k_{11}^2}{k_{00}^2}\right)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^2 - \frac{2}{\gamma^3}\left(4+6\gamma+\frac{k_{11}}{k_{00}}\right)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^3 + \frac{1}{\gamma^4}\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^4\right], \\
c' &= -\frac{1}{d'}\left[\frac{4}{\gamma^4}\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^4\right], \\
d' &= (-1+\gamma)^2\left[\left(1+\frac{k_{11}}{k_{00}}\right)^2 - \frac{2}{\gamma}\left(-1+2\gamma+\frac{k_{11}}{k_{00}}\right)\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right) + \frac{1}{\gamma^2}\left(\frac{k_{11}}{k_{00}}-\frac{k_{10}^2}{k_{00}^2}\right)^2\right].
\end{aligned}
\right\}
\tag{72}
$$

Judging from the form of the discriminant D given in Eq. (71), the value of p at which D becomes zero can be found by solving the sixth-order equation with an even degree other than the double root of $p = 0$. Therefore, we can apply Cardan's method in unmodified form.

The square root of the characteristic equation (59) is given in Eq. (70). If we express these as follows:

$$
\left.
\begin{aligned}
\lambda_1^2 &= -A\omega - B\omega^2 - \frac{a}{3}, \\
\lambda_2^2 &= -A\omega^2 - B\omega - \frac{a}{3}, \\
\lambda_3^2 &= -A - B - \frac{a}{3},
\end{aligned}
\right\}
\tag{73}
$$

we can then express the general solution by the linear combination of the fundamental solutions. For example,

$$
\Theta_{y0}(p, x_1) = \sum_{j=1}^{3}(C_j\,\mathrm{e}^{-\lambda_j x_1} + C_{j+3}\,\mathrm{e}^{\lambda_j x_1}). \tag{74}
$$

Substituting this into formulae (56), (55), and (53), we can express the other state quantities $W(p, x_1)$, $\Theta_{y1}(p, x_1)$, $\mathcal{Q}_{z0}(p, x_1)$, $\mathcal{M}_{y0}(p, x_1)$, and $\mathcal{M}_{y1}(p, x_1)$ with the linear combinations with constants C_1–C_6 and the fundamental solutions.

$$
W(p, x_1) = \sum_{j=1}^{3}\lambda_j\left(1 - \frac{\lambda_j^2}{p^2}\right)(-C_j\,\mathrm{e}^{-\lambda_j x_1} + C_{j+3}\,\mathrm{e}^{\lambda_j x_1}), \tag{75}
$$

$$
\Theta_{y1}(p, x_1) = -\frac{k_{00}}{k_{10}}\sum_{j=1}^{3}\left(1 + \gamma p^2 - \lambda_j^2(1+\gamma) + \frac{\lambda_j^4}{p^2}\right)(C_j\,\mathrm{e}^{-\lambda_j x_1} + C_{j+3}\,\mathrm{e}^{\lambda_j x_1}), \tag{76}
$$

$$
\mathcal{Q}_{z0}(p, x_1) = \sum_{j=1}^{3}(p^2 - \lambda_j^2)(C_j\,\mathrm{e}^{-\lambda_j x_1} + C_{j+3}\,\mathrm{e}^{\lambda_j x_1}), \tag{77}
$$

$$
\mathcal{M}_{y0}(p, x_1) = \sum_{j=1}^{3}\lambda_j(C_j\,\mathrm{e}^{-\lambda_j x_1} - C_{j+3}\,\mathrm{e}^{\lambda_j x_1}), \tag{78}
$$

$$
\mathcal{M}_{y1}(p, x_1) = \frac{k_{00}}{k_{10}}\sum_{j=1}^{3}\lambda_j\left(1 + \gamma p^2 - \lambda_j^2(1+\gamma) + \frac{\lambda_j^4}{p^2}\right)(-C_j\,\mathrm{e}^{-\lambda_j x_1} + C_{j+3}\,\mathrm{e}^{\lambda_j x_1}). \tag{79}
$$

4.2. Transfer matrix for a finite beam

By replacing the integral constant of the general solutions (74) and (75)–(79) with the state quantities at the starting edge of the beam, which has the dimensionless field length L_1, we can obtain the relation to the state quantities of the ending edge. The exponential functions are changed to hyperbolic functions. If we express the state quantities at the starting edge with subscript (0), the transfer function for the relation between the state quantities at the starting edge and those at beam position $x_1 = x_1$ can be expressed as follows.

$$\begin{bmatrix} W(p,x_1) \\ \Theta_{y0}(p,x_1) \\ \Theta_{y1}(p,x_1) \\ \mathscr{M}_{y1}(p,x_1) \\ \mathscr{M}_{y0}(p,x_1) \\ \mathscr{Q}_{z0}(p,x_1) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \\ t_{61} & t_{62} & t_{63} & t_{64} & t_{65} & t_{66} \end{bmatrix} \begin{bmatrix} W^{(0)} \\ \Theta_{y0}^{(0)} \\ \Theta_{y1}^{(0)} \\ \mathscr{M}_{y1}^{(0)} \\ \mathscr{M}_{y0}^{(0)} \\ \mathscr{Q}_{z0}^{(0)} \end{bmatrix}. \tag{80}$$

4.3. Transfer matrix for a semi-infinite beam

Here, we will give the transfer relation of a semi-infinite beam. First, we take the starting point of a beam on the origin of the position coordinate. Here, we consider that the beam extends infinitely toward the positive x direction. At the beam point of infinity, all the state quantities become zero. Thus, all exponential functions with positive exponents must be omitted. As a result, the terms of integral constants C_4–C_6 in the fundamental solutions become zero. By replacing the remaining integral constants C_1–C_3 with known state quantities of the beam starting edge, we can obtain the transfer relation for the semi-infinite beam.

If the starting point is the hinged support (Fig. 2(a)), the deflection and warping moments will be specified. Thus, if we denote these as $W^{(0)}$, $\mathscr{M}_{y0}^{(0)}$, and $\mathscr{M}_{y1}^{(0)}$, respectively, the transfer relation will be

$$\begin{bmatrix} W(p,x_1) \\ \Theta_{y0}(p,x_1) \\ \Theta_{y1}(p,x_1) \\ \mathscr{M}_{y1}(p,x_1) \\ \mathscr{M}_{y0}(p,x_1) \\ \mathscr{Q}_{z0}(p,x_1) \end{bmatrix} = \begin{bmatrix} t_{11} & t_{14} & t_{15} \\ t_{21} & t_{24} & t_{25} \\ t_{31} & t_{34} & t_{35} \\ t_{41} & t_{44} & t_{45} \\ t_{51} & t_{54} & t_{55} \\ t_{61} & t_{64} & t_{65} \end{bmatrix} \begin{bmatrix} W^{(0)} \\ \mathscr{M}_{y1}^{(0)} \\ \mathscr{M}_{y0}^{(0)} \end{bmatrix}. \tag{81}$$

If the starting point is the slider (Fig. 2(b)), two types of sectional rotation, and shearing force are specified. Thus, if we denote these as $\Theta_{y0}^{(0)}$, $\Theta_{y1}^{(0)}$, and $\mathscr{Q}_{z0}^{(0)}$, respectively, the transfer relation will be

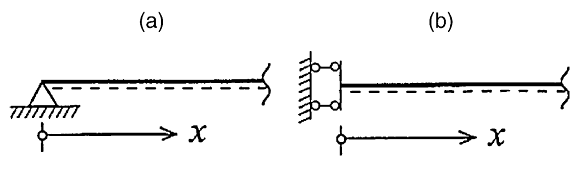

Fig. 2. Boundary conditions of starting edge for the semi-infinite beam.

$$\begin{bmatrix} W(p,x_1) \\ \Theta_{y0}(p,x_1) \\ \Theta_{y1}(p,x_1) \\ \mathscr{M}_{y1}(p,x_1) \\ \mathscr{M}_{y0}(p,x_1) \\ \mathscr{Q}_{z0}(p,x_1) \end{bmatrix} = \begin{bmatrix} t_{12} & t_{13} & t_{16} \\ t_{22} & t_{23} & t_{26} \\ t_{32} & t_{33} & t_{36} \\ t_{42} & t_{43} & t_{46} \\ t_{52} & t_{53} & t_{56} \\ t_{62} & t_{63} & t_{66} \end{bmatrix} \begin{bmatrix} \Theta_{y0}^{(0)} \\ \Theta_{y1}^{(0)} \\ \mathscr{Q}_{z0}^{(0)} \end{bmatrix}. \tag{82}$$

The elements of the field transfer matrix for a semi-infinite beam are given in Appendix A.

5. Problem setting

In relation to the six types of state functions that exist at the starting edge of a semi-infinite beam, or at $x = 0$, we will consider the following six types of actions. Namely, any parameter having a nonzero value will be applied to only one state quantity, and the two remaining specified state quantities must be zero.

- Problem 1: Constant deflection velocity $\dot{w}_1^{(0)}$; however, the two types of warping moments must be zero ($\overline{M}_{y1}^{(0)} = \overline{M}_{y0}^{(0)} = 0$).
- Problem 2: Constant sectional rotation velocity at the zeroth step $\dot{\theta}_{y0}^{(0)}$; however, the sectional rotation velocity at the first step and the shearing load must be zero ($\dot{\theta}_{y1}^{(0)} = \overline{Q}_{z0}^{(0)} = 0$).
- Problem 3: Constant sectional rotation velocity at the first step $\dot{\theta}_{y1}^{(0)}$; however, the sectional rotation velocity at the zeroth step and the shearing load must be zero ($\dot{\theta}_{y0}^{(0)} = \overline{Q}_{z0}^{(0)} = 0$).
- Problem 4: Constant dimensionless warping moment at the first step $\overline{M}_{y1}^{(0)}$; however, the deflection velocity and usual bending moment must be zero ($\dot{w}_1^{(0)} = \overline{M}_{y0}^{(0)} = 0$).
- Problem 5: Constant dimensionless bending moment $\overline{M}_{y0}^{(0)}$; however, the deflection velocity and warping moment at the first step must be zero ($\dot{w}_1^{(0)} = \overline{M}_{y1}^{(0)} = 0$).
- Problem 6: Constant dimensionless shearing load $\overline{Q}_{z0}^{(0)}$; however, the two types of sectional rotation velocity must be zero ($\dot{\theta}_{y0}^{(0)} = \dot{\theta}_{y1}^{(0)} = 0$).

In the problems above, the sign $\dot{()}$ indicates differentiation with respect to dimensionless time t_1. The action forces of the starting edge, $\overline{M}_{y0}^{(0)}$, $\overline{M}_{y1}^{(0)}$, and $\overline{Q}_{z0}^{(0)}$, conform to the nondimensional definition, Eq. (49). Problems 1, 4, and 5 are conditions for the hinged support (Fig. 2(a)). Problems 2, 3, and 6 are conditions for the slider support (Fig. 2(b)). When these boundary conditions are converted, based on the definition of Laplace transform formula (51), the following are provided.

$$\begin{array}{lllllllll}
\text{Problem 1:} & W^{(0)} & = & \dot{w}_1^{(0)}/p^2, & \mathscr{M}_{y0}^{(0)} & = & 0, & \mathscr{M}_{y1}^{(0)} & = & 0, \\
\text{Problem 2:} & \Theta_{y0}^{(0)} & = & \dot{\theta}_{y0}^{(0)}/p^2, & \Theta_{y1}^{(0)} & = & 0, & \mathscr{Q}_{z0}^{(0)} & = & 0, \\
\text{Problem 3:} & \Theta_{y0}^{(0)} & = & 0, & \Theta_{y1}^{(0)} & = & \dot{\theta}_{y1}^{(0)}/p^2, & \mathscr{Q}_{z0}^{(0)} & = & 0, \\
\text{Problem 4:} & W^{(0)} & = & 0, & \mathscr{M}_{y1}^{(0)} & = & \overline{M}_{y1}/p, & \mathscr{M}_{y0}^{(0)} & = & 0, \\
\text{Problem 5:} & W^{(0)} & = & 0, & \mathscr{M}_{y1}^{(0)} & = & 0, & \mathscr{M}_{y0}^{(0)} & = & \overline{M}_{y0}^{(0)}/p, \\
\text{Problem 6:} & \Theta_{y0}^{(0)} & = & 0, & \Theta_{y1}^{(0)} & = & 0, & \mathscr{Q}_{z0}^{(0)} & = & \overline{Q}_{z0}^{(0)}/p.
\end{array}$$

These values are to be substituted into the specified state quantity vector for the beam starting edge on the right hand-sides of formulae (81) and (82).

6. Laplace inverse transform

6.1. Integral expression for the inverse transform

After the specified boundary values are defined and the inverse transform is taken, the solution of all state quantities in the time domain can be obtained. Typically, the function for which the specified boundary values are given can be represented as

$$F(p,x_1) = \sum_{j=1}^{3} F_j(p)\mathrm{e}^{-\lambda_j x_1}. \tag{83}$$

$F_j(p)$ on the right-hand side will be the function for which the specified transformed value of the starting edge is multiplied by each element of the transfer matrix t_{ij}. In other words, it is the function for which the element t_{ij} is multiplied by zero, or $1/p$ or $1/p^2$, depending on the specified values of the "Problem."

Taking the limit as $|p| \to \infty$ for function $F_j(p)$, we obtain

$$\lim_{|p|\to\infty} F_j(p) = 0 \quad (j = 1,2,3). \tag{84}$$

Additionally, in some cases the denominator or numerator of the function $F_j(p)$ is or is not multiplied by λ_j, and this is clear from the form of the function of the transfer matrix element.

When we calculate the limit of $|p| \to \infty$ of the three types of roots λ_1, λ_2, and λ_3 obtained from the characteristic equation (59), the results are as follows:

$$\left.\begin{aligned} &\lim_{|p|\to\infty} \frac{\lambda_1}{p\gamma^{1/2}} = 1,\\ &\lim_{|p|\to\infty} \frac{\lambda_2}{p} = 1,\\ &\lim_{|p|\to\infty} \frac{\lambda_3}{p} = 1. \end{aligned}\right\} \tag{85}$$

Also, we can easily prove that λ_1 and λ_2 will each have a factor of $p^{1/2}$. The formula is shown below.

$$\left.\begin{aligned} &\lim_{|p|\to 0} \frac{\lambda_1}{p^{1/2}} = \pm(-\mathrm{i})^{1/2} = \pm\frac{1}{\sqrt{2}}(1-\mathrm{i}),\\ &\lim_{|p|\to 0} \frac{\lambda_2}{p^{1/2}} = \pm(+\mathrm{i})^{1/2} = \pm\frac{1}{\sqrt{2}}(1+\mathrm{i}),\\ &\lim_{|p|\to 0} \lambda_3 = \pm\frac{1}{\gamma^{1/2}}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right)^{1/2}. \end{aligned}\right\} \tag{86}$$

Fig. 3 shows variations in these three types of roots on the real axis and on the imaginary axis.

The Laplace inverse transform assumes the following form.

$$f(t_1,x_1) = \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F(p,x_1)\mathrm{e}^{pt_1}\,\mathrm{d}p. \tag{87}$$

Namely, according to formula (83), the following equation is obtained.

$$f(t_1,x_1) = \sum_{j=1}^{3} \frac{1}{2\pi\mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_j(p)\mathrm{e}^{(pt_1-\lambda_j x_1)}\,\mathrm{d}p, \tag{88}$$

where 'i' is an imaginary unit; i.e., $\mathrm{i} = \sqrt{-1}$. In addition, the constant c is a value set such that all singular points of function $F(p,x_1)$ on the complex plane p are on the left side of the line $p = c$, which is parallel to

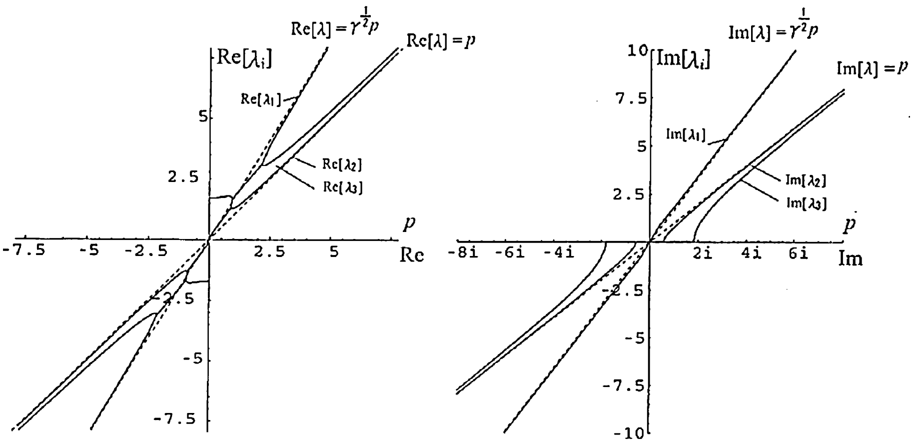

Fig. 3. Changes in the characteristic roots on the real axis and the imaginary axis.

the imaginary axis. The segment in the positive direction (upward) of $p = c$, which is parallel to this imaginary axis, is what we would like to call the Integral path Br_1 after the Bromwich integral.

If we take Laplace inverse transform while keeping all the state quantities in vector form, we obtain the following equation:

$$\begin{bmatrix} \dot{w}_1(t_1,x_1) \\ \dot{\theta}_{y0}(t_1,x_1) \\ \dot{\theta}_{y1}(t_1,x_1) \\ \overline{M}_{y1}(t_1,x_1) \\ \overline{M}_{y0}(t_1,x_1) \\ \overline{Q}_{z0}(t_1,x_1) \end{bmatrix} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \begin{bmatrix} \bar{t}_{11} & \bar{t}_{12} & \bar{t}_{13} & \bar{t}_{14} & \bar{t}_{15} & \bar{t}_{16} \\ \bar{t}_{21} & \bar{t}_{22} & \bar{t}_{23} & \bar{t}_{24} & \bar{t}_{25} & \bar{t}_{26} \\ \bar{t}_{31} & \bar{t}_{32} & \bar{t}_{33} & \bar{t}_{34} & \bar{t}_{35} & \bar{t}_{36} \\ \bar{t}_{41} & \bar{t}_{42} & \bar{t}_{43} & \bar{t}_{44} & \bar{t}_{45} & \bar{t}_{46} \\ \bar{t}_{51} & \bar{t}_{52} & \bar{t}_{53} & \bar{t}_{54} & \bar{t}_{55} & \bar{t}_{56} \\ \bar{t}_{61} & \bar{t}_{62} & \bar{t}_{63} & \bar{t}_{64} & \bar{t}_{65} & \bar{t}_{66} \end{bmatrix} \mathrm{e}^{pt_1}\, \mathrm{d}p \cdot \begin{bmatrix} \dot{w}_1^{(0)} \\ \dot{\theta}_{y0}^{(0)} \\ \dot{\theta}_{y1}^{(0)} \\ \overline{M}_{y1}^{(0)} \\ \overline{M}_{y0}^{(0)} \\ \overline{Q}_{z0}^{(0)} \end{bmatrix}. \tag{89}$$

The elements of transfer matrix t_{ij} in the rows of deformations $w_1(t_1,x_1)$, $\theta_{y0}(t_1,x_1)$, and $\theta_{y1}(t_1,x_1)$ are differentiated by dimensionless time t_1 and then multiplied by p. In addition, because of the coefficient of the state quantity of the deformations for the starting edge, the elements will be multiplied by $1/p^2$. Because of the coefficient of the state quantity of the stress resultants, the elements will also be multiplied by $1/p$. The elements of transfer matrix t_{ij} in the rows of stress resultants $M_{y1}(t_1,x_1)$, $M_{y0}(t_1,x_1)$, and $Q_{z0}(t_1,x_1)$ are multiplied only by $1/p^2$, corresponding to the coefficient of the state quantity of the deformations for the starting edge, and by $1/p$, corresponding to the coefficient of the stress resultants. This is expressed symbolically in Table 1.

Table 1
Multipliers of transfer elements

	$\frac{1}{p^2}\times$: $\Downarrow$	$\frac{1}{p}\times$: $\Downarrow$
Time integrated: $p\times$: $\Rightarrow$	$\frac{1}{p}\times \boldsymbol{t}_{vv}$	$1\times \boldsymbol{t}_{vf}$
	$\frac{1}{p^2}\times \boldsymbol{t}_{fv}$	$\frac{1}{p}\times \boldsymbol{t}_{ff}$

From this operation, we can understand that the symmetry of transfer matrix element t_{ij} with respect to the subsidiary diagonal line is retained. The newly-obtained transfer matrix element $\bar{t}_{ij}$ is shown in Appendix B.

6.2. Branch points and poles

The factors that constitute the elements of the transfer matrix for the finite beam and the semi-infinite beam are as follows.

$$\lambda_j, p, p^2$$
$$\lambda_j^2 - \lambda_{j+1}^2$$
$$\lambda_j^2 - p^2$$
$$\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2$$
$$p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)$$
$$-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)$$

Branch points exist at positions where the cubic root λ_j becomes zero or where the discriminant D, which exists in the square root that is a factor of the cubic root, becomes zero. The proposed solution for the former is the value p, which satisfies the equation below, according to the sixth-degree algebraic equation (59).

$$c = 0. \tag{90}$$

Setting the right-hand side of formula (58c) to zero, we can then solve the following.

$$p^2\left[\frac{1}{\gamma}\left(\frac{k_{11}}{k_{00}} - \frac{k_{10}^2}{k_{00}^2}\right) + p^2\left(1 + \frac{k_{11}}{k_{00}}\right) + p^4\gamma\right] = 0. \tag{91}$$

According to the solution obtained by use of the quadratic formula, we can obtain five kinds of roots.

$$\left.\begin{aligned} &p = 0 \quad \text{(Double root)},\\ &p = \pm \mathrm{i}\sqrt{\frac{1}{2\gamma}\left[\left(1+\frac{k_{11}}{k_{00}}\right) \pm \sqrt{\left(1-\frac{k_{11}}{k_{00}}\right)^2 + 4\frac{k_{10}^2}{k_{00}^2}}\right]}. \end{aligned}\right\} \tag{92}$$

Four roots, or all roots except the double root $p = 0$, become purely imaginary numbers, and they are distributed symmetrically on the imaginary axis with respect to the origin.

With reference to Fig. 3, the following conclusions are obtained.

- At $p = 0$, characteristic roots λ_1 and λ_2 become zero. Characteristic root λ_1 does not have a branch point on the imaginary axis.
- Characteristic root λ_2 has branch points $\pm\mathrm{i}\, p_{i2}$, which are symmetrical with respect to the origin on the imaginary axis, where,

$$p_{i2} \equiv \sqrt{\frac{1}{2\gamma}\left[\left(1+\frac{k_{11}}{k_{00}}\right) - \sqrt{\left(1-\frac{k_{11}}{k_{00}}\right)^2 + 4\frac{k_{10}^2}{k_{00}^2}}\right]}.$$

- Characteristic root λ_3 has branch points $\pm\mathrm{i}\, p_{i3}$, which are symmetrical with respect to the origin on the imaginary axis, where,

$$p_{i3} \equiv \sqrt{\frac{1}{2\gamma}\left[\left(1+\frac{k_{11}}{k_{00}}\right)+\sqrt{\left(1-\frac{k_{11}}{k_{00}}\right)^2+4\frac{k_{10}^2}{k_{00}^2}}\right]}.$$

Next, we will solve for the value p, for which the discriminant D becomes zero. The equation for which the discriminant (71) is set to zero is as follows.

$$d'p^2(p^6+a'p^4+b'p^2+c')=0. \tag{93}$$

Therefore, by substituting zero for each factor, we obtain

$$\left.\begin{array}{l} p^2=0, \\ p^6+a'p^4+b'p^2+c'=0. \end{array}\right\} \tag{94}$$

As mentioned previously, Cardan's formula can be applied to the root of this sixth-degree algebraic equation (94b). The result is as follows

$$p^2=-A'-B'-\frac{a'}{3},\quad -A'\omega-B'\omega^2-\frac{a'}{3},\quad -A'\omega^2-B'\omega-\frac{a'}{3}, \tag{95}$$

where

$$\left.\begin{array}{l} A'=\left[\frac{1}{2}(q^{*'}+\sqrt{q^{*'2}+4p^{*'3}})\right]^{1/3}, \\ B'=\left[\frac{1}{2}(q^{*'}-\sqrt{q^{*'2}+4p^{*'3}})\right]^{1/3}, \\ A'B'=-p^{*'}. \end{array}\right\} \tag{96}$$

$p^{*'}$ and $q^{*'}$ can be calculated from

$$\left.\begin{array}{l} p^{*'}=\frac{1}{3}\left(b'-\frac{a'^2}{3}\right), \\ q^{*'}=c'-\frac{1}{3}a'b'+\frac{2}{27}a'^3. \end{array}\right\} \tag{97}$$

If Eq. (95); i.e. the square of the root of an equation for which the discriminant is set to zero, is expressed in descending order, as shown below,

$$p^2=0^2,\quad p_1^2,\quad p_2^2,\quad p_3^2.$$

Therefore, all three roots are real numbers, since the discriminant in Eq. (96),

$$D'\equiv q^{*'2}+4p^{*'3}, \tag{98}$$

becomes negative. This is known as the "*casus irreducibilis*". Furthermore, from the relation between the roots and the coefficients in formula (71), i.e.,

$$a'=-(p_1^2+p_2^2+p_3^2)\leqslant 0,\quad b'=p_1^2p_2^2+p_2^2p_3^2+p_3^2p_1^2\geqslant 0,\quad c'=-p_1^2p_2^2p_3^2\leqslant 0,$$

we can prove that all three roots in the cubic equation become positive real numbers (Fig. 4). As a result, all the roots of the eight-degree algebraic equation (93) become

$$p=0\ \text{(double root)},\quad \pm p_1,\quad \pm p_2,\quad \pm p_3.$$

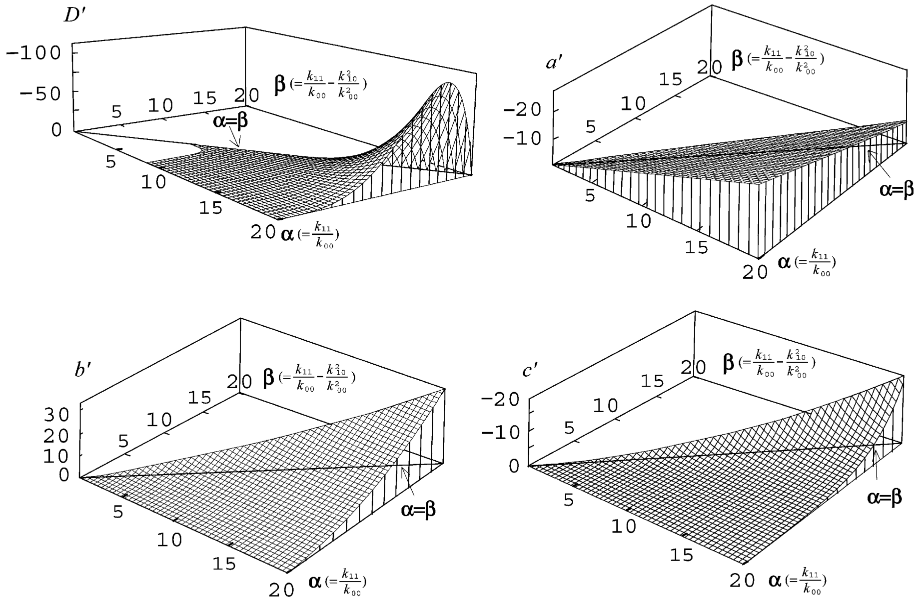

Fig. 4. Positive and negative of D', a', b' and c'.

Thus, the roots are distributed symmetrically on the real axis with respect to the origin.

The proposed solution for the pole is considered to be the p-value that satisfies the following equations, because of the form of the factors which constitute the denominator of the fraction for the transfer matrix element.

$$\left.\begin{array}{l}\lambda_j = 0 \quad (j = 1\text{–}3),\\ \lambda_j^2 - \lambda_{j+1}^2 = 0 \quad (j = 1\text{–}3),\\ p^{\alpha} = 0 \quad (\alpha = 1/2, 1, 3/2, \ldots).\end{array}\right\} \tag{99}$$

The first equation, (99a), specifies the position where λ_j becomes zero. Thus, it corresponds to the position of the branch point, as calculated earlier; namely, the point which satisfies this first condition is the branch point on the imaginary axis and is also the pole of order 1/2. In order to determine the value p which satisfies the second equation (99b), we substituted Eq. (73) into this. In the end, the result is identical with the position at which the discriminant D becomes zero, as calculated previously. Namely, the point which satisfies this second condition is the branch point on the real axis and is also the pole of order 1/2. The value p that satisfies the third equation (99c) is the origin on the complex plane p and also the branch point, and can be said to be the pole of order α.

In summary, the branch points and poles for function $F_j(p)$ match the zero points of the characteristic root λ_j and the zero points of its discriminant D. Therefore, the integration of function $F_j(p)$ will be satisfactory if we concentrate on only the branch points of characteristic root λ_j.

6.3. Branch cut

As indicated in Fig. 5, we let the branch cut go towards the origin from the left side on the negative real axis. If a branch point exists on the imaginary axis, we draw the line such that it goes around the branch point from the origin. These paths between adjacent branch points can be identified as follows. Let the positive direction of the real axis correspond to the sequence L_0, L_1, L_2, L_3, L_4, L_5, L_6, and let the negative

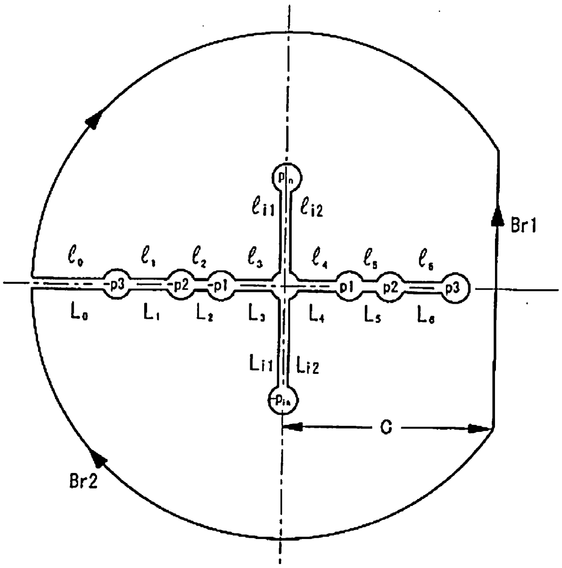

Fig. 5. Branch cut.

direction to which the line returns around the branch point $+p_3$ on the right edge of real axis correspond to the sequence l_6, l_5, l_4, l_3, l_2, l_1, l_0. The imaginary axis downward from the origin will be L_{i1}. The section in which the line extends toward the origin after going around the branch point of the negative imaginary numbers will be L_{i2}. Upward from the origin will be l_{i2}, and the section in which the line extends toward the origin after going around the branch points of positive imaginary numbers will be denoted as l_{i1}.

Because we introduced the branch cut on the real axis, the value of $\sqrt{D}$ has to be bifurcated into positive and negative from the branch point at which the discriminant D becomes zero. Because of this, the symbols A and B in Cardan's formula (68) have to be mutually exchanged. In the end, two conjugate complex solutions are interchangeable and located above and below the branch cut; thereby, the real root remains a real root.

When we introduce a branch cut on the imaginary axis, the characteristic root, λ_2, for instance, will be divided into positive and negative values from branch point $\pm ip_{i2}$, at which λ_2 becomes zero. Thus, we introduce the following.

$$\lambda_2(p) = \mu_2\sqrt{(p - ip_{i2})(p + ip_{i2})}.$$

The length from the origin to p is denoted as ρ. Then, we calculate the above equation. Considering the change in the phase angle, the results are as follows:

$$\text{Section } L_{i1}: \ \mu_2\sqrt{(\rho e^{-\frac{\pi}{2}i} - ip_{i2})(\rho e^{-\frac{\pi}{2}i} + ip_{i2})} = \mu_2(p_{i2}^2 - \rho^2)^{1/2} e^{-\pi i} = -\lambda_2$$

$$\text{Section } L_{i2}: \ \mu_2\sqrt{(\rho e^{-\frac{\pi}{2}i} - ip_{i2})(\rho e^{-\frac{\pi}{2}i} + ip_{i2})} = \mu_2(p_{i2}^2 - \rho^2)^{1/2} = \lambda_2$$

$$\text{Section } l_{i2}: \ \mu_2\sqrt{(\rho e^{\frac{\pi}{2}i} - ip_{i2})(\rho e^{\frac{\pi}{2}i} + ip_{i2})} = \mu_2(p_{i2}^2 - \rho^2)^{1/2} = \lambda_2$$

$$\text{Section } l_{i1}: \ \mu_2\sqrt{(\rho e^{\frac{\pi}{2}i} - ip_{i2})(\rho e^{\frac{\pi}{2}i} + ip_{i2})} = \mu_2(p_{i2}^2 - \rho^2)^{1/2} e^{\pi i} = -\lambda_2$$

With the characteristic root λ_3, similar sign-switching occurs at each path-section.

The change in the characteristic root on the real axis can also be determined in the same manner, by changing the discriminant D into the factorization below with positive real number d'.

$$D \equiv -\frac{d'}{27}p^2(p^2 - p_1^2)(p^2 - p_2^2)(p^2 - p_3^2).$$

In Fig. 6, we show the changes in these three kinds of characteristic roots along the cutting line on the real axis. Fig. 7 shows changes in the characteristic roots along the cutting line on the imaginary axis.

Table 2 provides the changes in the characteristic line for each section.

6.4. Paths of integration

Based on our previous discussion on branch points and poles, here we would like to choose the four kinds of paths of integration shown in Fig. 8, in order to obtain the Laplace inverse transform of the obtained function $F(p, x_1)$.

Table 3 shows the integral paths classified by differences in branch points of the integrands and differences in the sign of the exponent in the exponential functions.

Fig. 8(a) shows a case where an integral path Br_2 of a circular arc shape can be taken at the right side of $p = c$ where no branch points and poles exist. When Eq. (88) and the Cauchy integral are taken into consideration, the following equations hold.

$$\begin{aligned}
I_1 &= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_1(p) \mathrm{e}^{p\left(t_1 - \frac{\lambda_1}{p} x_1\right)} \, \mathrm{d}p \\
&= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_1(p) \mathrm{e}^{p\left(t_1 - \gamma^{1/2} x_1\right)} \, \mathrm{d}p, \\
I_2 &= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_2(p) \mathrm{e}^{p\left(t_1 - \frac{\lambda_2}{p} x_1\right)} \, \mathrm{d}p \\
&= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_2(p) \mathrm{e}^{p(t_1 - x_1)} \, \mathrm{d}p, \\
I_3 &= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_3(p) \mathrm{e}^{p\left(t_1 - \frac{\lambda_3}{p} x_1\right)} \, \mathrm{d}p \\
&= \lim_{|p|\to\infty} \int_{\mathrm{Br}_2} F_3(p) \mathrm{e}^{p(t_1 - x_1)} \, \mathrm{d}p.
\end{aligned}$$

According to Eq. (84) and Jordan's lemma, the following equations hold.

$$I_1 = 0 \quad (t_1 < \gamma^{1/2} x_1), \quad I_2 = 0 \quad (t_1 < x_1), \quad I_3 = 0 \quad (t_1 < x_1). \tag{100}$$

By substituting the nondimensional variables (47) into the above equation, we can rewrite it as a usual expression in terms of time t and space x, as follows:

$$\left.\begin{aligned}
I_1 &= 0 \quad (c_2 t < x), \\
I_2 &= 0 \quad (c_1 t < x), \\
I_3 &= 0 \quad (c_1 t < x).
\end{aligned}\right\} \tag{101}$$

Namely, among three kinds of waves caused by impacts, the integral term of function $F_1(p)$ propagates at the velocity of transversal wave c_2 and the integral terms of functions $F_2(p)$ and $F_3(p)$ propagate at the velocity of longitudinal wave c_1.

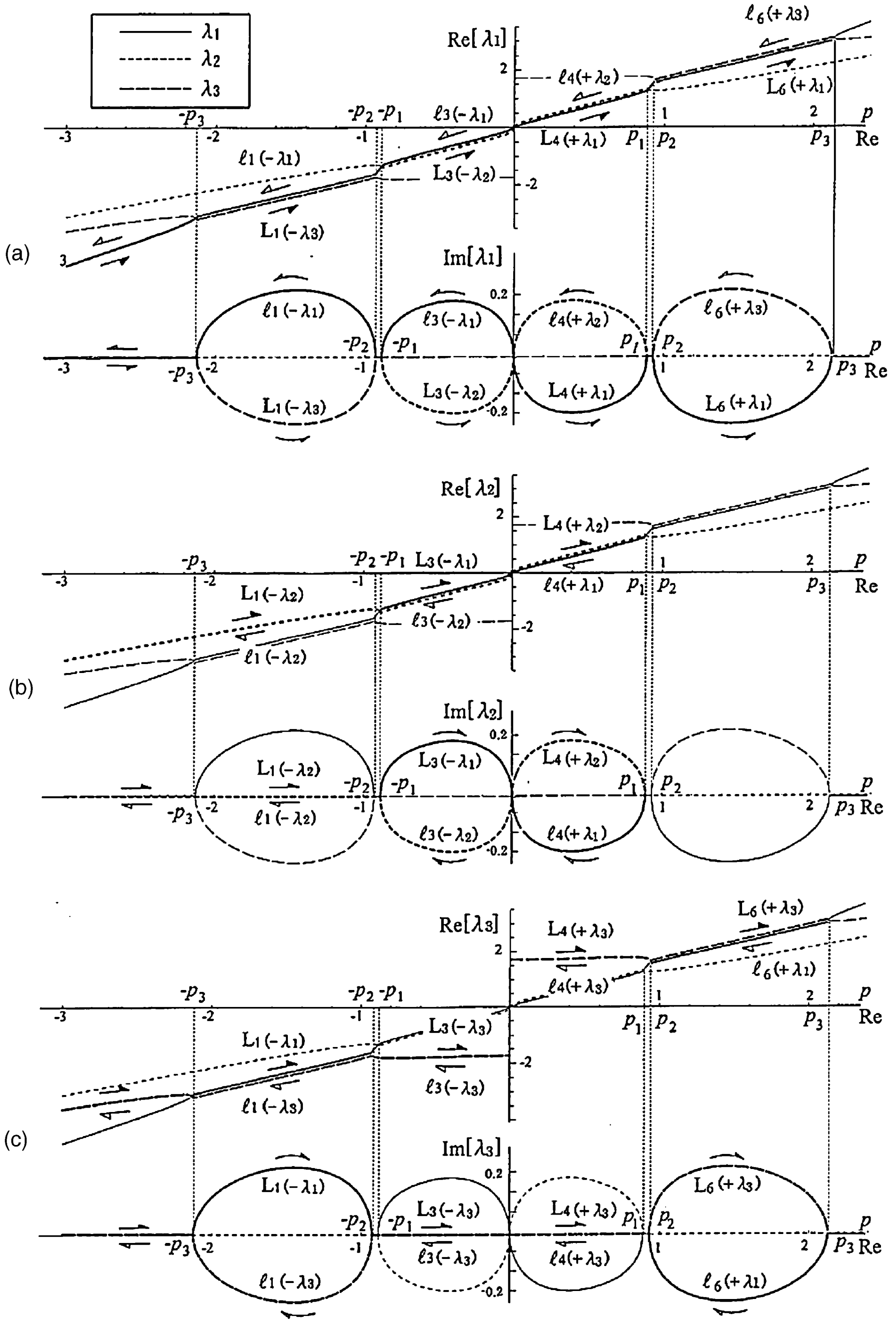

Fig. 6. Changes in the characteristic roots on the real axis.

In order to integrate the function $F_1(p)$ over the range $\gamma^{1/2}x_1 < t_1$, the integral section must bypass to the left side section of segment Br_1, where the exponent of the exponential function becomes negative. In that section, branch points exist only on the real axis; thus, we can take the integral path as shown in Fig. 8(b).

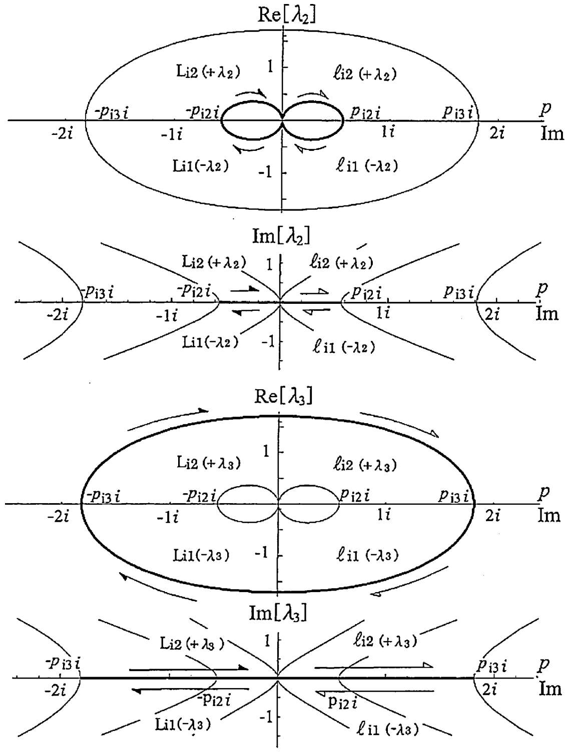

Fig. 7. Changes in the characteristic roots on the imaginary axis.

Table 2
Changes in characteristic roots

Section	$\vec{L}_0$	$\vec{L}_1$	$\vec{L}_2$	$\vec{L}_3$	$\vec{L}_4$	$\vec{L}_5$	$\vec{L}_6$
p	$-\rho$	$-\rho$	$-\rho$	$-\rho$	ρ	ρ	ρ
(λ_1)	$-\lambda_1$	$-\lambda_3$	$-\lambda_1$	$-\lambda_2$	λ_1	λ_1	λ_1
(λ_2)	$-\lambda_2$	$-\lambda_2$	$-\lambda_2$	$-\lambda_1$	λ_2	λ_2	λ_2
(λ_3)	$-\lambda_3$	$-\lambda_1$	$-\lambda_3$	$-\lambda_3$	λ_3	λ_3	λ_3
Section	$\overleftarrow{l}_0$	$\overleftarrow{l}_1$	$\overleftarrow{l}_2$	$\overleftarrow{l}_3$	$\overleftarrow{l}_4$	$\overleftarrow{l}_5$	$\overleftarrow{l}_6$
p	$-\rho$	$-\rho$	$-\rho$	$-\rho$	ρ	ρ	ρ
(λ_1)	$-\lambda_1$	$-\lambda_1$	$-\lambda_1$	$-\lambda_1$	λ_2	λ_1	λ_3
(λ_2)	$-\lambda_2$	$-\lambda_2$	$-\lambda_2$	$-\lambda_2$	λ_1	λ_2	λ_2
(λ_3)	$-\lambda_3$	$-\lambda_3$	$-\lambda_3$	$-\lambda_3$	λ_3	λ_3	λ_1
Section	$\downarrow L_{i1}$	$\uparrow L_{i2}$	$\uparrow l_{i2}$	$\downarrow l_{i1}$			
p	$-\mathrm{i}\rho$	$-\mathrm{i}\rho$	$\mathrm{i}\rho$	$\mathrm{i}\rho$			
(λ_2)	$-\lambda_2$	λ_2	λ_2	$-\lambda_2$			
(λ_3)	$-\lambda_3$	λ_3	λ_3	$-\lambda_3$			

Likewise, in order to integrate the function $F_2(p)$ over the range $x_1 < t_1$, we can take the integral path shown in Fig. 8(c); concerning the integral of function $F_3(p)$, we can take the integral path shown in Fig. 8(d).

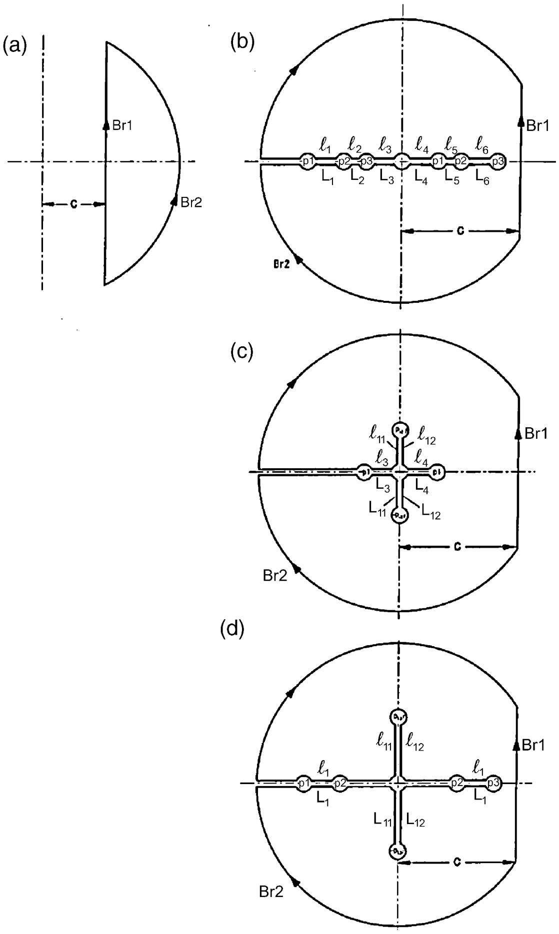

Fig. 8. Four kinds of integral paths.

Table 3
Integral paths

Integration	$t_1 < x_1$	$x_1 < t_1 < \gamma^{1/2}x_1$	$\gamma^{1/2}x_1 < t_1$
$\int_{Br_2} F_1(p)e^{pt_1-\lambda_1 x_1}\,dp$	Fig. 8(a)	Fig. 8(a)	Fig. 8(b)
$\int_{Br_2} F_2(p)e^{pt_1-\lambda_2 x_1}\,dp$	Fig. 8(a)	Fig. 8(c)	Fig. 8(c)
$\int_{Br_2} F_3(p)e^{pt_1-\lambda_3 x_1}\,dp$	Fig. 8(a)	Fig. 8(d)	Fig. 8(d)

In Table 2, which summarizes the changes of a characteristic root, when characteristic roots of the same sign appear above and below (or to the left and right sides of) the branch cut, they cancel each other and do not contribute to the integral value. Therefore, the sections where contribution to the integrated value takes place for each integrand are as listed below.

$$F_1(p):\ L_1 + L_3 + L_4 + L_6 + l_6 + l_4 + l_3 + l_1$$
$$F_2(p):\ L_3 + L_4 + l_4 + l_3$$
$$F_3(p):\ L_1 + L_6 + l_6 + l_1$$

Furthermore, the number of appearances of $\pm\lambda_j$ ($j = 1$–3) for each section on the real axis is always once above and once below the branch cut (Table 2); thus, all integral values in this section cancel each other out and in the end their total becomes zero.

In Table 2, the sign always reverses when the branch cut on the imaginary axis is crossed from left to right or vice versa. Thus, contribution to the integral value takes place in all sections of the branch cut on the imaginary axis.

6.5. Path integral

At first glance, integrand $F_j(p)$ ($j = 1$–3) seems complicated; however, as shown in Table 2, the terms in λ_j^2 ($j = 1$–3) transmute while cycling through the roles; thus, the integrand $F_j(p)$ ($j = 1$–3) itself transmutes while circulating. However, when factors λ_j and p exist in the integrand as the first order, the sign change in this factor directly affects the change in the sign of the integrand. We would like to explain this mechanism by reference to the following examples.

For instance, let us consider a case in which the factors λ_j and p do not exist as terms of the first order. The term of $j = 2$ in the transfer matrix element $\bar{t}_{51}$ of the infinite beam will be as follows:

$$F_2(p)\mathrm{e}^{-\lambda_2 x_1} \equiv \frac{\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}\mathrm{e}^{-\lambda_2 x_1}.$$

Therefore, neither factor λ_j ($j = 1, 2, 3$) nor p exists as a first order term. The inverse transform on the real axis can be calculated as

$$\begin{aligned}\int_{L_3+L_4+l_4+l_3} F_2(p)\mathrm{e}^{-\lambda_2 x_1 + pt_1}\,\mathrm{d}p &= \int_{p_1}^{0} F_1\,\mathrm{e}^{\lambda_1 x_1 - \rho t_1}\,\mathrm{d}(-\rho) + \int_0^{p_1} F_2\,\mathrm{e}^{-\lambda_2 x_1 + \rho t_1}\,\mathrm{d}\rho + \int_{p_1}^{0} F_1\,\mathrm{e}^{-\lambda_1 x_1 + \rho t_1}\,\mathrm{d}\rho \\ &\quad + \int_0^{p_1} F_2\,\mathrm{e}^{\lambda_2 x_1 - \rho t_1}\,\mathrm{d}(-\rho) \\ &= 2\int_0^{p_1} \{F_1 \sinh(\lambda_1 x_1 - \rho t_1) - F_2 \sinh(\lambda_2 x_1 - \rho t_1)\}\,\mathrm{d}\rho.\end{aligned}$$

The inverse transform of the same element on the imaginary axis can be calculated as

$$\begin{aligned}\int_{L_{i1}+L_{i2}+l_{i2}+l_{i1}} F_2(p)\mathrm{e}^{-\lambda_2 x_1 + pt_1}\,\mathrm{d}p &= \int_0^{p_{i2}} F_2\,\mathrm{e}^{\lambda_2 x_1 - \mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho) + \int_{p_{i2}}^{0} F_2\,\mathrm{e}^{-\lambda_2 x_1 - \mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho) \\ &\quad + \int_0^{p_{im}} F_2\,\mathrm{e}^{-\lambda_2 x_1 + \mathrm{i}\rho t_1}\,d(\mathrm{i}\rho) + \int_{p_{i2}}^{0} F_2\,\mathrm{e}^{\lambda_2 x_1 + \mathrm{i}\rho t_1}\,\mathrm{d}(\mathrm{i}\rho) \\ &= -\int_0^{p_{i2}} \mathrm{i}F_2(p)\left[(\mathrm{e}^{\lambda_2 x_1} - \mathrm{e}^{-\lambda_2 x_1})(\mathrm{e}^{\mathrm{i}\rho t_1} + \mathrm{e}^{-\mathrm{i}\rho t_1})\right]\mathrm{d}\rho \\ &= -4\mathrm{i}\int_0^{p_{i2}} F_2(p)\sinh\lambda_2 x_1 \cos\rho t_1\,\mathrm{d}\rho.\end{aligned}$$

Likewise, the inverse transform of $F_3(p)$ can be obtained from the term $j = 3$ in the transfer matrix element $\bar{t}_{51}$.

Next, let us consider the case in which the factor λ_j exists as a first order term and p does not exist as a first order term. The $j = 2$ term in the transfer matrix element $\bar{t}_{61}$ of the infinite beam will be

$$F_2(p)\mathrm{e}^{-\lambda_2 x_1} \equiv -\frac{(\lambda_2^2 - p^2)[\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2]}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}\mathrm{e}^{-\lambda_2 x_1}.$$

Thus, only factor λ_2 exists as a first order term. Its inverse transform on the real axis can be calculated as

$$\int_{L_3+L_4+l_4+l_3} F_2(p)\mathrm{e}^{-\lambda_2 x_1+pt_1}\,\mathrm{d}p = \int_{p_1}^{0} -F_1\,\mathrm{e}^{\lambda_1 x_1-\rho t_1}\,\mathrm{d}(-\rho) + \int_{0}^{p_1} F_2\,\mathrm{e}^{-\lambda_2 x_1+\rho t_1}\,\mathrm{d}\rho + \int_{p_1}^{0} F_1\,\mathrm{e}^{-\lambda_1 x_1+\rho t_1}\,\mathrm{d}\rho$$

$$+ \int_{0}^{p_1} -F_2\,\mathrm{e}^{\lambda_2 x_1-\rho t_1}\,\mathrm{d}(-\rho)$$

$$= -2\int_{0}^{p_1} \{F_1\cosh(\lambda_1 x_1-\rho t_1) - F_2\cosh(\lambda_2 x_1-\rho t_1)\}\,\mathrm{d}\rho.$$

The inverse transform of the same element on the imaginary axis can be calculated as

$$\int_{L_{i1}+L_{i2}+l_{i2}+l_{i1}} F_2(p)\mathrm{e}^{-\lambda_2 x_1+pt_1}\,\mathrm{d}p = \int_{0}^{p_{i2}} -F_2\,\mathrm{e}^{\lambda_2 x_1-\mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho) + \int_{p_{i2}}^{0} F_2\,\mathrm{e}^{-\lambda_2 x_1-\mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho)$$

$$+ \int_{0}^{p_{i2}} F_2\,\mathrm{e}^{-\lambda_2 x_1+\mathrm{i}\rho t_1}\,\mathrm{d}(\mathrm{i}\rho) + \int_{p_{i2}}^{0} -F_2\,\mathrm{e}^{\lambda_2 x_1+\mathrm{i}\rho t_1}\,\mathrm{d}(\mathrm{i}\rho)$$

$$= \int_{0}^{p_{i2}} \mathrm{i}F_2(p)(\mathrm{e}^{\lambda_2 x_1} + \mathrm{e}^{-\lambda_2 x_1})(\mathrm{e}^{\mathrm{i}\rho t_1} + \mathrm{e}^{-\mathrm{i}\rho t_1})\,\mathrm{d}\rho$$

$$= 4\mathrm{i}\int_{0}^{p_{i2}} F_2(p)\cosh\lambda_2 x_1\cos\rho t_1\,\mathrm{d}\rho.$$

Likewise, the inverse transform of $F_3(p)$ can be obtained from the term $j=3$ in the transfer matrix element $\bar{t}_{61}$.

Now, let us consider the case in which the factor p exists as a first order term and λ_j does not exist as a first order term, which is the opposite of the previous case. The term $j=2$ in the transfer matrix element $\bar{t}_{11}$ of the infinite beam will be

$$F_2(p)\mathrm{e}^{-\lambda_2 x_1} \equiv \frac{(\lambda_2^2-p^2)[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\mathrm{e}^{-\lambda_2 x_1}.$$

Thus, only factor p exists as a first order term. Its inverse transform on the real axis can be calculated as

$$\int_{L_3+L_4+l_4+l_3} F_2(p)\mathrm{e}^{-\lambda_2 x_1+pt_1}\,\mathrm{d}p = \int_{p_1}^{0} -F_1\,\mathrm{e}^{\lambda_1 x_1-\rho t_1}\,\mathrm{d}(-\rho) + \int_{0}^{p_1} F_2\,\mathrm{e}^{-\lambda_2 x_1+\rho t_1}\,\mathrm{d}\rho + \int_{p_1}^{0} F_1\,\mathrm{e}^{-\lambda_1 x_1+\rho t_1}\,\mathrm{d}\rho$$

$$+ \int_{0}^{p_1} -F_2\,\mathrm{e}^{\lambda_2 x_1-\rho t_1}\,\mathrm{d}(-\rho)$$

$$= -2\int_{0}^{p_1} \{F_1\cosh(\lambda_1 x_1-\rho t_1) - F_2\cosh(\lambda_2 x_1-\rho t_1)\}\,\mathrm{d}\rho.$$

The inverse transform of the same element on the imaginary axis can be calculated as

$$\begin{aligned}\int_{L_{i1}+L_{i2}+l_{i2}+l_{i1}} F_2(p)\mathrm{e}^{-\lambda_2 x_1+pt_1}\,\mathrm{d}p &= \int_0^{p_{i2}} -F_2\,\mathrm{e}^{\lambda_2 x_1-\mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho) + \int_{p_{i2}}^{0} -F_2\,\mathrm{e}^{-\lambda_2 x_1-\mathrm{i}\rho t_1}\,\mathrm{d}(-\mathrm{i}\rho)\\ &\quad + \int_0^{p_{i2}} F_2\,\mathrm{e}^{-\lambda_2 x_1+\mathrm{i}\rho t_1}\,\mathrm{d}(\mathrm{i}\rho) + \int_{p_{i2}}^{0} F_2\,\mathrm{e}^{\lambda_2 x_1+\mathrm{i}\rho t_1}\,\mathrm{d}(\mathrm{i}\rho)\\ &= -\int_0^{p_{i2}} \mathrm{i}F_2(p)(\mathrm{e}^{\lambda_2 x_1}-\mathrm{e}^{-\lambda_2 x_1})(\mathrm{e}^{\mathrm{i}\rho t_1}-\mathrm{e}^{-\mathrm{i}\rho t_1})\,\mathrm{d}\rho\\ &= 4\int_0^{p_{i2}} F_2(p)\sinh\lambda_2 x_1 \sin\rho t_1\,\mathrm{d}\rho.\end{aligned}$$

Likewise, the inverse transform of $F_3(p)$ can be obtained from the term $j=3$ in the transfer matrix element $\bar{t}_{11}$.

6.6. Integral around a pole

According to the impact problem involving the beam model, as previously stated, the poles on the complex plane overlap the branch points. Among these, other than the origin, the branch-pole points will be of the order 1/2. Thus, even if these are integrated, they will not have nonzero values, as shown below. In contrast, the branch-pole point on the origin will have nonzero values, depending on the order.

As an example of a pole of order 1/2, we will take the case of $j=1$ in the transfer matrix element $\bar{t}_{21}$.

$$F_1(p)\mathrm{e}^{-\lambda_1 x_1} \equiv \frac{p[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\mathrm{e}^{-\lambda_1 x_1}.$$

A factor of the numerator p and an inside factor p in the factor $(\lambda_1^2-\lambda_2^2)$ of the denominator cancel each other out. Because of the existence of λ_1 in the denominator which has $p^{1/2}$ as a factor, the function retains a pole of order 1/2. We transform this function into a contour integral for infinitesimal radius r, which encloses the origin of polar coordinate $p=r\mathrm{e}^{\mathrm{i}\theta}$, and we then take the limit as $r\to 0$. The constant coefficient can be calculated from Eq. (86) and is thus omitted. Taking out the pole of order 1/2 only, we calculate

$$\lim_{|p|\to 0}\oint_r \frac{1}{p^{1/2}}\,\mathrm{d}p = \lim_{r\to 0}\int_{-\pi}^{\pi}\frac{\mathrm{i}r\mathrm{e}^{\mathrm{i}\theta}}{(r\mathrm{e}^{\mathrm{i}\theta})^{1/2}}\,\mathrm{d}\theta = \lim_{r\to 0}\int_{-\pi}^{\pi}\mathrm{i}r^{1/2}\mathrm{e}^{\mathrm{i}\theta/2}\,\mathrm{d}\theta = 0.$$

This shows that the integral value around the origin will be zero.

As an example in which the pole of order 1 is not affected by the branch, we will consider a case where $j=1$ in the transfer matrix element $\bar{t}_{11}$.

$$F_1(p)\mathrm{e}^{-\lambda_1 x_1} \equiv \frac{(\lambda_1^2-p^2)[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{p(\lambda_3^2-\lambda_2^2)(\lambda_1^2-\lambda_2^2)}\mathrm{e}^{-\lambda_1 x_1}.$$

An inside factor p in the numerator $(\lambda_1^2-p^2)$ and an inside factor p in the denominator $(\lambda_1^2-\lambda_2^2)$ cancel each other out. In the end, this function has a pole of order 1 because of the factor p at the left end of denominator. When we take the inverse transform, the circle line integral around the origin with infinitesimal radius r must be executed. From the calculus of residues

$$\frac{1}{2\pi\mathrm{i}}\oint_r F_1(p)\mathrm{e}^{-\lambda_1 x_1+pt_1}\,\mathrm{d}p = \mathrm{Res}\,[F_1(p)\mathrm{e}^{-\lambda_1 x_1+pt_1};0] = \lim_{|p|\to 0}(p-0)F_1(p)\mathrm{e}^{-\lambda_1 x_1+pt_1} = \lim_{|p|\to 0}\frac{\lambda_1^2}{\lambda_1^2-\lambda_2^2} = \frac{1}{2}$$

with the relation from Eq. (86)

Table 4
Changes in the characteristic root around the origin

Quadrant	III	IV	I	II
(λ_1)	$-\lambda_2$	λ_1	λ_2	$-\lambda_1$
(λ_2)	$-\lambda_1$	λ_2	λ_1	$-\lambda_2$

$$\lim_{|p|\to 0}\frac{\lambda_2^2}{\lambda_1^2} = -1.$$

Although, as shown in Table 4, the characteristic roots λ_1 and λ_2 change their roles around the origin (Fig. 9) for each quadrant, this relation holds unchangeably and is not affected by the branch.

As an example in which the pole of order 1 is affected by the branch, we will consider a case where $j = 1$ in the transfer matrix element $\bar{t}_{51}$ concerning bending moment.

$$F_1(p)e^{-\lambda_1 x_1} \equiv \frac{[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 x_1}.$$

Factors in the numerator $[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]$ and denominator $(\lambda_3^2 - \lambda_1^2)$ include λ_3^2. Thus, at the origin it will become nonzero. However, the factor $(\lambda_1^2 - \lambda_2^2)$ has p as an inside factor; therefore, at $p = 0$ it has a pole of order 1. The contour integral above is obtained by calculus of residues as

$$\frac{1}{2\pi i}\oint_r F_1(p)e^{-\lambda_1 x_1 + pt_1}\,dp = \mathrm{Res}[F_1(p)e^{-\lambda_1 x_1 + pt_1}; 0] = \lim_{|p|\to 0}(p-0)F_1(p)e^{-\lambda_1 x_1 + pt_1} = \lim_{|p|\to 0}\frac{p}{\lambda_1^2 - \lambda_2^2}.$$

Calculating the denominator $\lambda_1^2 - \lambda_2^2$ from Table 4 and Eq. (86), we obtain

$$\lambda_1^2 - \lambda_2^2 = \begin{cases} +2ip & \text{(quadrants I, III)}, \\ -2ip & \text{(quadrants II, IV)}. \end{cases}$$

Therefore, the residue theorem cannot be applied. In this case, the integrand also changes.

$$\lim_{|p|\to 0} F_1(p)e^{-\lambda_1 x_1 + pt_1} = \lim_{|p|\to 0}\frac{[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 x_1 + pt_1} = \begin{cases} +\frac{1}{2i}\lim_{|p|\to 0}\frac{1}{p} & \text{(quadrants I, III)}, \\ -\frac{1}{2i}\lim_{|p|\to 0}\frac{1}{p} & \text{(quadrants II, IV)}. \end{cases}$$

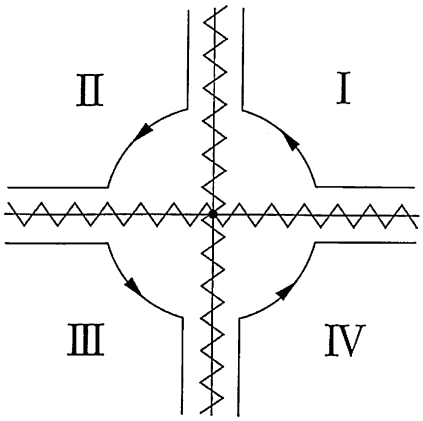

Fig. 9. Integral path around the origin.

By introducing the polar coordinate $p = re^{i\theta}$,

$$\lim_{|p|\to 0} \frac{1}{2\pi i} \oint_r F_1(p) e^{-\lambda_1 x_1 + p t_1} \, dp = \frac{1}{2\pi i} \int_{-\pi}^{-\frac{\pi}{2}} \frac{1}{2ire^{i\theta}} ire^{i\theta} \, d\theta + \frac{1}{2\pi i} \int_{-\frac{\pi}{2}}^{0} \left(-\frac{1}{2ire^{i\theta}} \right) ire^{i\theta} \, d\theta$$

$$+ \frac{1}{2\pi i} \int_{0}^{\frac{\pi}{2}} \frac{1}{2ire^{i\theta}} ire^{i\theta} \, d\theta + \frac{1}{2\pi i} \int_{\frac{\pi}{2}}^{\pi} \left(-\frac{1}{2ire^{i\theta}} \right) ire^{i\theta} \, d\theta$$

$$= \frac{1}{8i} - \frac{1}{8i} + \frac{1}{8i} - \frac{1}{8i} = 0.$$

We can see that there is no contribution to the integral value.

As an example in which the pole is of the order 3/2, we consider a case where $j = 1$ in the transfer matrix element $\bar{t}_{12}$.

$$F_1(p) e^{-\lambda_1 x_1} \equiv \frac{\lambda_1 [\lambda_2^2 + \lambda_3^2 - (1+\gamma) p^2]}{p(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 x_1}.$$

The numerator expression λ_1 has the inside factor $p^{1/2}$, and both the denominator expression $(\lambda_1^2 - \lambda_2^2)$ and the denominator expression p have p. In the end, this integrand $F_1(p)e^{-\lambda_1 x_1}$ will have a pole of order 3/2. The residue theorem can be used only for a pole of an order expressed by an integer. Thus, we can differentiate this integrand until it becomes a function with a pole of the first order and then take back the integral from its residue. For this function, first order differentiation is sufficient.

$$\frac{1}{2\pi i} \oint_r F_1(p) e^{-\lambda_1 x_1 + p t_1} \, dp = \int_0^{x_1} \frac{1}{2\pi i} \oint_r \frac{\partial}{\partial x_1} F_1(p) e^{-\lambda_1 x_1 + p t_1} \, dp \, dx_1 = \int_0^{x_1} \operatorname{Res}\left[\frac{\partial}{\partial x_1} F_1(p) e^{-\lambda_1 x_1 + p t_1}; 0 \right] dx_1.$$

Here, from calculus of residues, we obtain

$$\operatorname{Res}\left[\frac{\partial}{\partial x_1} F_1(p) e^{-\lambda_1 x_1 + p t_1}; 0 \right] = \lim_{|p|\to 0} (p - 0) \frac{\partial}{\partial x_1} F_1(p) e^{-\lambda_1 x_1 + p t_1} = \lim_{|p|\to 0} p \frac{-\lambda_1^2 [\lambda_2^2 + \lambda_3^2 - (1+\gamma) p^2]}{p(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} e^{-\lambda_1 x_1 + p t_1}$$

$$= \lim_{|p|\to 0} \frac{\lambda_1^2}{\lambda_1^2 - \lambda_2^2} = \frac{1}{2}.$$

Thus, in the end, we obtain

$$\frac{1}{2\pi i} \oint_r F_1(p) e^{-\lambda_1 x_1 + p t_1} \, dp = \int_0^{x_1} \operatorname{Res}\left[\frac{\partial}{\partial x_1} F_1(p) e^{-\lambda_1 x_1 + p t_1}; 0 \right] dx_1 = \int_0^{x_1} \frac{1}{2} \, dx_1 = \frac{x_1}{2}.$$

Functions of orders other than this, regardless of whether they are affected by a branch, can be treated in the same manner.

6.7. Solutions of problems

The inverse transformation required to obtain the quantities listed was carried out as outlined in the previous sections; the final expression is

Table 5
Elements T_{ij} of transfer matrix in three time ranges

	Problem 1	Problem 2	Problem 3	Problem 4	Problem 5	Problem 6
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{11}^R + I_{11}^I + \frac{1}{2}$	$I_{12}^R + I_{12}^I + \frac{x_1}{2}$	$I_{13}^R + I_{13}^I$	$I_{14}^R + I_{14}^I$	$I_{15}^R + I_{15}^I$	$I_{16}^R + I_{16}^I$
$\gamma^{1/2}x_1 < t_1$	$I_{11}^I + 1$	$I_{12}^I + x_1$	I_{13}^I	I_{14}^I	I_{15}^I	I_{16}^I
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{21}^R + I_{21}^I$	$I_{22}^R + I_{22}^I + \frac{1}{2}$	$I_{23}^R + I_{23}^I$	$I_{24}^R + I_{24}^I$	$I_{25}^R + I_{25}^I$	$I_{26}^R + I_{26}^I$
$\gamma^{1/2}x_1 < t_1$	I_{21}^I	$I_{22}^I + 1$	I_{23}^I	I_{24}^I	I_{25}^I	I_{26}^I
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{31}^R + I_{31}^I$	$I_{32}^R + I_{32}^I$	$I_{33}^R + I_{33}^I$	$I_{34}^R + I_{34}^I$	$I_{35}^R + I_{35}^I$	$I_{36}^R + I_{36}^I$
$\gamma^{1/2}x_1 < t_1$	I_{31}^I	I_{32}^I	I_{33}^I	I_{34}^I	I_{35}^I	I_{36}^I
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{41}^R + I_{41}^I$	$I_{42}^R + I_{42}^I$	$I_{43}^R + I_{43}^I$	$I_{44}^R + I_{44}^I$	$I_{45}^R + I_{45}^I$	$I_{46}^R + I_{46}^I$
$\gamma^{1/2}x_1 < t_1$	I_{41}^I	I_{42}^I	I_{43}^I	I_{44}^I	I_{45}^I	I_{46}^I
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{51}^R + I_{51}^I$	$I_{52}^R + I_{52}^I$	$I_{53}^R + I_{53}^I$	$I_{54}^R + I_{54}^I$	$I_{55}^R + I_{55}^I + \frac{1}{2}$	$I_{56}^R + I_{56}^I + \frac{x_1}{2}$
$\gamma^{1/2}x_1 < t_1$	I_{51}^I	I_{52}^I	I_{53}^I	I_{54}^I	$I_{55}^I + 1$	$I_{56}^I + x_1$
$t_1 < x_1$	0	0	0	0	0	0
$x_1 < t_1 < \gamma^{1/2}x_1$	$I_{61}^R + I_{61}^I$	$I_{62}^R + I_{62}^I$	$I_{63}^R + I_{63}^I$	$I_{64}^R + I_{64}^I$	$I_{65}^R + I_{65}^I$	$I_{66}^R + I_{66}^I + \frac{1}{2}$
$\gamma^{1/2}x_1 < t_1$	I_{61}^I	I_{62}^I	I_{63}^I	I_{64}^I	I_{65}^I	$I_{66}^I + 1$

$$\begin{bmatrix} \dot{w}_1(t_1,x_1) \\ \dot{\theta}_{y0}(t_1,x_1) \\ \dot{\theta}_{y1}(t_1,x_1) \\ \overline{M}_{y1}(t_1,x_1) \\ \overline{M}_{y0}(t_1,x_1) \\ \overline{Q}_{z0}(t_1,x_1) \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} & T_{13} & T_{14} & T_{15} & T_{16} \\ T_{21} & T_{22} & T_{23} & T_{24} & T_{25} & T_{26} \\ T_{31} & T_{32} & T_{33} & T_{34} & T_{35} & T_{36} \\ T_{41} & T_{42} & T_{43} & T_{44} & T_{45} & T_{46} \\ T_{51} & T_{52} & T_{53} & T_{54} & T_{55} & T_{56} \\ T_{61} & T_{62} & T_{63} & T_{64} & T_{65} & T_{66} \end{bmatrix} \begin{bmatrix} \dot{w}_1^{(0)} \\ \dot{\theta}_{y0}^{(0)} \\ \dot{\theta}_{y1}^{(0)} \\ \overline{M}_{y1}^{(0)} \\ \overline{M}_{y0}^{(0)} \\ \overline{Q}_{z0}^{(0)} \end{bmatrix}. \tag{102}$$

Elements T_{ij} $(i,j = 1,2,\ldots,6)$ of the transfer matrix in the three time ranges are given in Table 5. Again, we can recognize the reciprocal relation through the array symmetry with respect to the subsidiary diagonal line. Symbols for Table 5 are given in Appendix C.

7. Transverse impact behavior for a semi-infinite beam

Six kinds of transverse impact mentioned previously were applied to the origin ($x_1 = 0$) of the x axis of a semi-infinite elastic beam, which has a rectangular cross-section of width b and height h. Unit warping functions and unit shearing functions at the zeroth step and the first step are as follows (see Fig. 10).

$$Z_0(z) = z,$$

$$Z_1(z) = -\frac{\sqrt{21}}{4}h\left[\left(\frac{z}{h/2}\right) - \frac{5}{3}\left(\frac{z}{h/2}\right)^3\right],$$

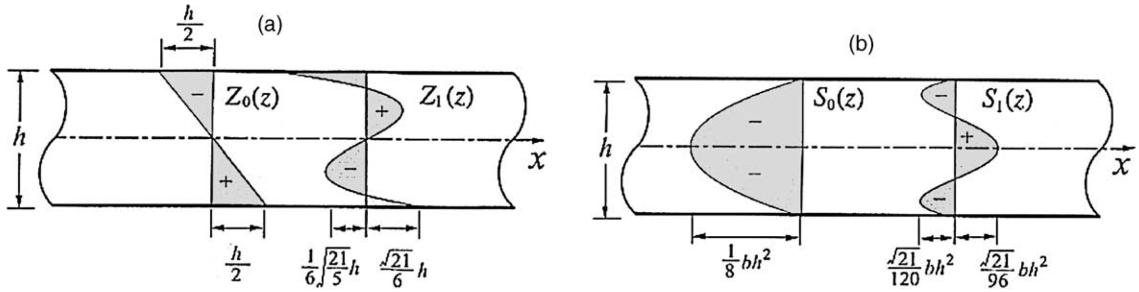

Fig. 10. Unit warping function and unit shearing function.

$$S_0(z) = -\frac{1}{8}bh^2\left[1-\left(\frac{z}{h/2}\right)^2\right],$$

$$S_1(z) = \frac{\sqrt{21}}{96}bh^2\left[1-6\left(\frac{z}{h/2}\right)^2+5\left(\frac{z}{h/2}\right)^4\right].$$

From this, the sectional area of beam A, the warping resistance matrix $\boldsymbol{F}$, the shearing resistance matrix $\boldsymbol{R}$, and the shear correction matrix $\boldsymbol{k}'$ can be calculated as follows:

$$A = bh,$$

$$\boldsymbol{F} = \frac{1}{12}bh^3\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$\boldsymbol{R} = bh^5\begin{bmatrix}\dfrac{1}{120} & -\dfrac{1}{240\sqrt{21}}\\ -\dfrac{1}{240\sqrt{21}} & \dfrac{1}{1080}\end{bmatrix},$$

$$\boldsymbol{R}^{-1} = \frac{1}{bh^5}\begin{bmatrix}\dfrac{672}{5} & \dfrac{144\sqrt{21}}{5}\\ \dfrac{144\sqrt{21}}{5} & \dfrac{6048}{5}\end{bmatrix},$$

$$\boldsymbol{k}' = \begin{bmatrix}\dfrac{14}{15} & \dfrac{\sqrt{21}}{5}\\ \dfrac{\sqrt{21}}{5} & \dfrac{42}{5}\end{bmatrix}.$$

Poisson's ratio v is set to

$$v = 0.29.$$

The following results represent the case in which six kinds of loadings are all considered "step (force) action". When we let the dimensionless time at impact be $t_1 = 0$, the distributions of the state quantities in the beam axis at dimensionless time $t_1 = 5$ are indicated in the order of problem numbers shown in Fig. 11. This figure shows a distribution in the axial direction for each state quantity with the amount of transverse

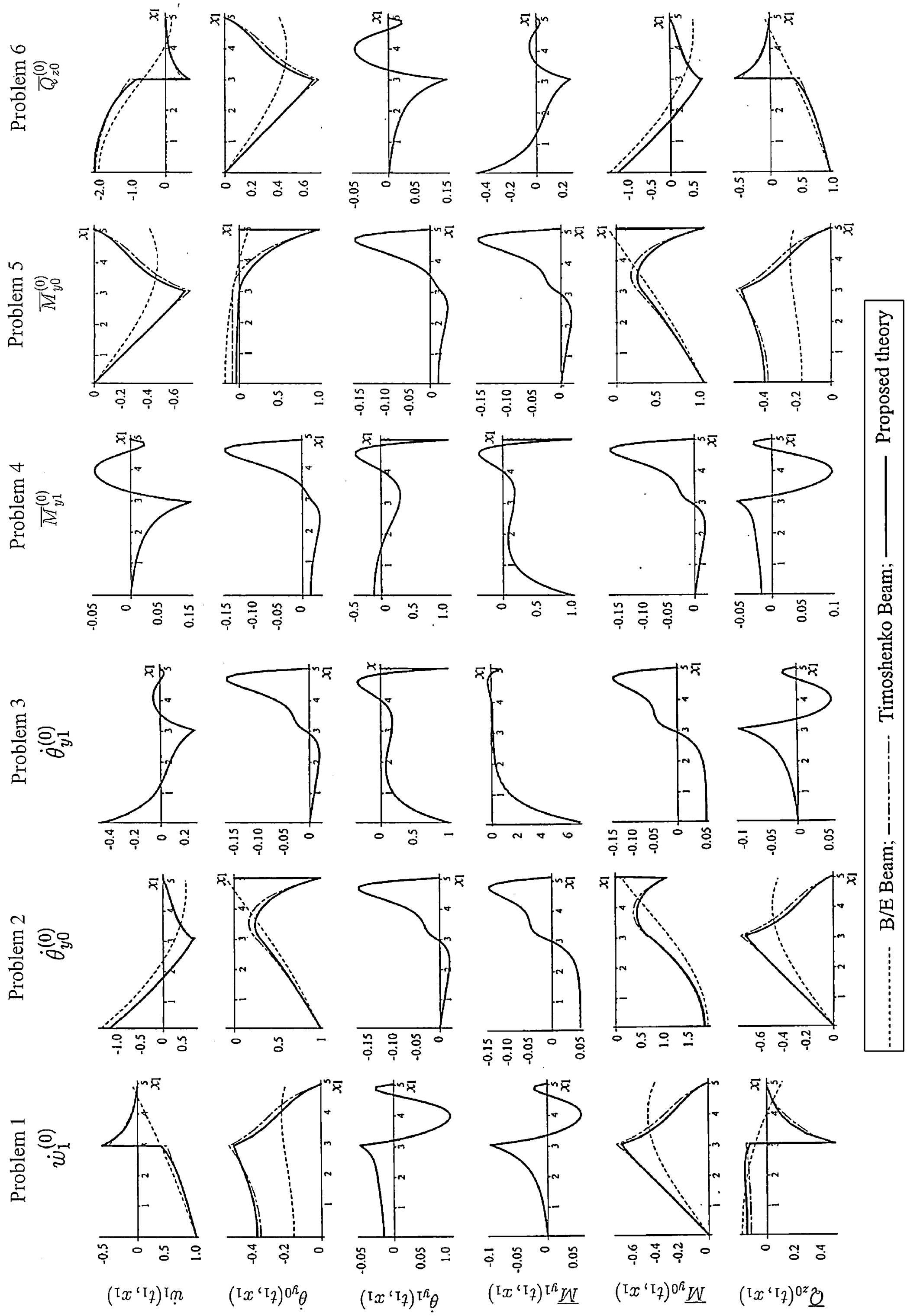

Fig. 11. Distribution of each state quantity in axial direction at dimensionless time $t_1 = 5$.

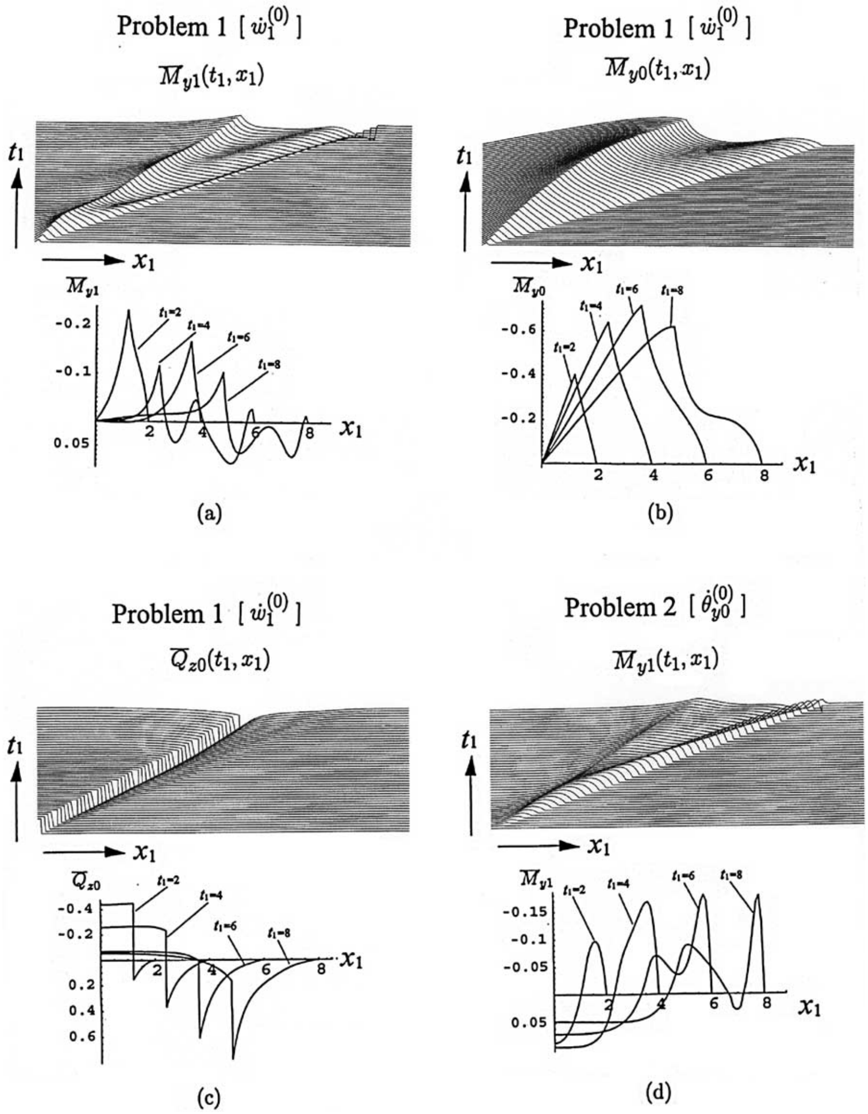

Fig. 12. (a) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\dot{w}_1^{(0)}$. (b) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\dot{w}_1^{(0)}$. (c) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\dot{w}_1^{(0)}$. (d) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\dot{\theta}_{y0}^{(0)}$. (e) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\dot{\theta}_{y0}^{(0)}$. (f) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\dot{\theta}_{y0}^{(0)}$. (g) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\dot{\theta}_{y1}^{(0)}$. (h) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\dot{\theta}_{y1}^{(0)}$. (i) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\dot{\theta}_{y1}^{(0)}$. (j) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\overline{M}_{y1}^{(0)}$. (k) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\overline{M}_{y1}^{(0)}$. (l) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\overline{M}_{y1}^{(0)}$. (m) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\overline{M}_{y0}^{(0)}$. (n) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\overline{M}_{y0}^{(0)}$. (o) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\overline{M}_{y0}^{(0)}$. (p) Wave propagation of warping moment $\overline{M}_{y1}$ at the action of $\overline{Q}_{z0}^{(0)}$. (q) Wave propagation of bending moment $\overline{M}_{y0}$ at the action of $\overline{Q}_{z0}^{(0)}$. (r) Wave propagation of shear force $\overline{Q}_{z0}$ at the action of $\overline{Q}_{z0}^{(0)}$.

impact, shown under the line of each problem, being the only unit amount. Items are indicated in the same row and column order employed in the transfer matrix element $\bar{t}_{ij}$ (Appendix B). Therefore, a so-called

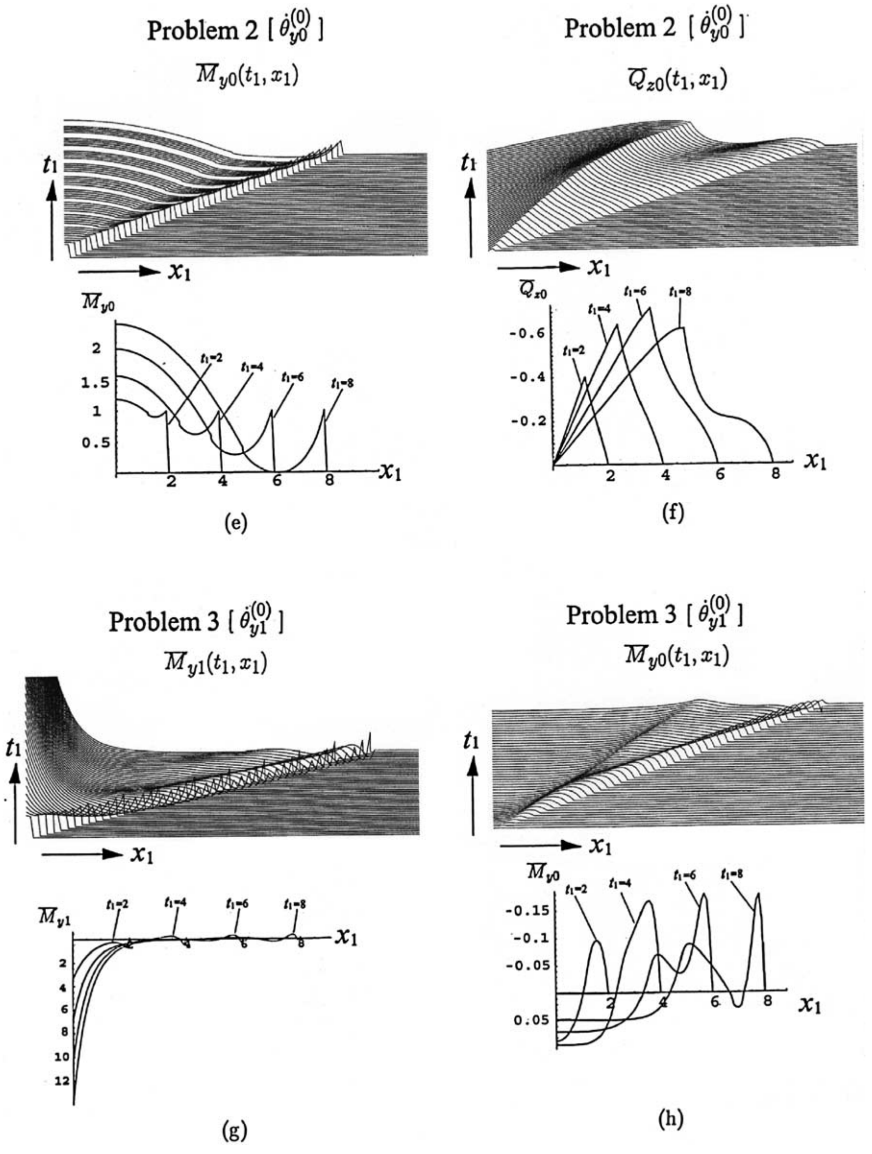

Fig. 12 (*continued*)

"reciprocal relation" is observed about the subsidiary diagonal line. This reciprocal relation always holds, regardless of time.

Let us compare the calculation result from our theory (solid line) with that from the Bernoulli/Euler Beam theory (dotted line) and that from the Timoshenko Beam theory (broken line). The B/E beam does not include a longitudinal wave, so it turns out to be a single smooth transit line without discontinuity. The Timoshenko beam includes a transverse wave and a longitudinal wave; thus, a clear discontinuity exists at the point at which they separate. Although similar to the solution obtained from the Timoshenko beam theory, the solution obtained from our theory deviates slightly because we overlapped two kinds of

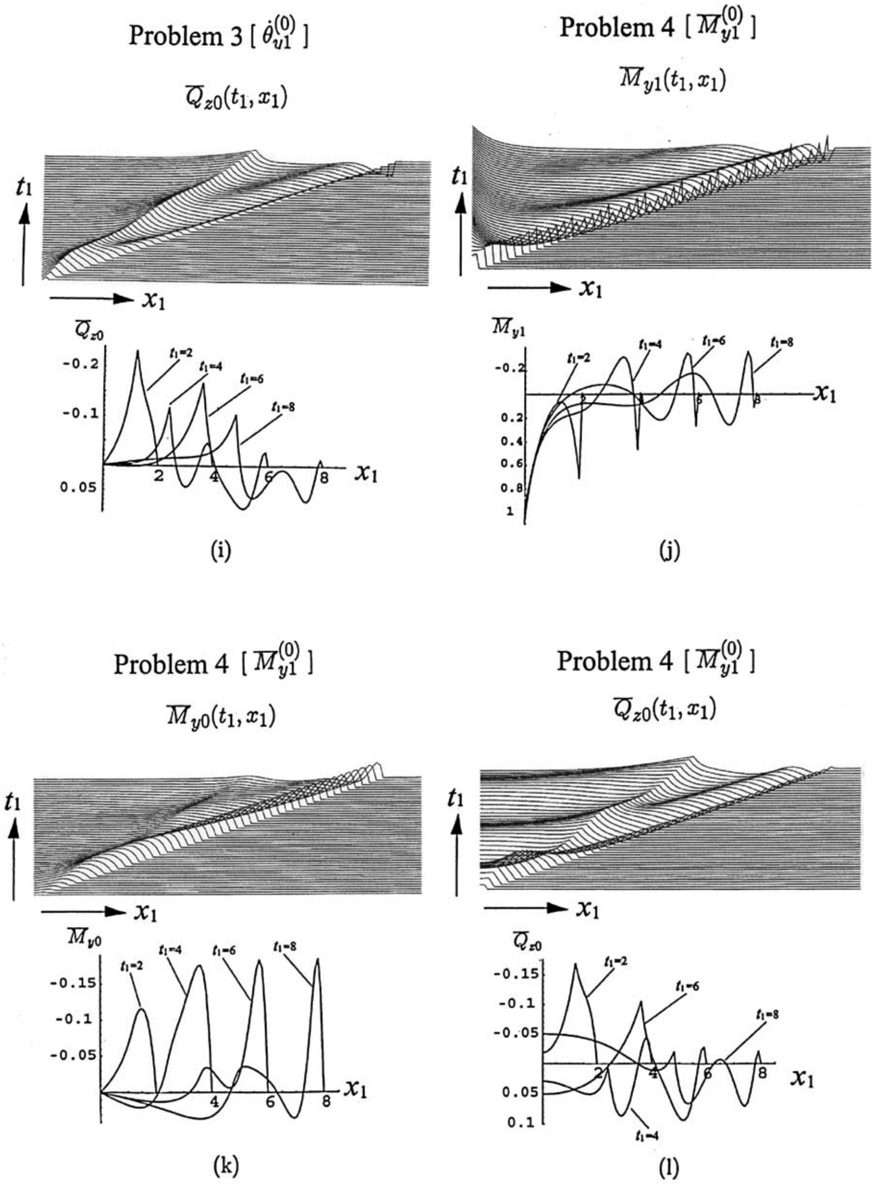

Fig. 12 (*continued*)

longitudinal waves of warping. The difference between the result obtained from the Timoshenko beam theory (broken line) and that obtained from our proposed higher-order beam theory (solid line) lies in the effect of nonlinear warping. Also, the rotation of cross section $\dot{\theta}_{y1}(t_1, x_1)$ and its warping moment $\overline{M}_{y1}(t_1, x_1)$ for nonlinear warping in the first step, indicated in the third and fourth lines and the third and fourth columns of Fig. 11, represent the newly solved distribution for the higher-order beam theory. Deformation and its corresponding stress resultants, except at the area of the starting edge, are almost equally distributed. Distributions of these shapes show a more radical change between positive and negative than do the

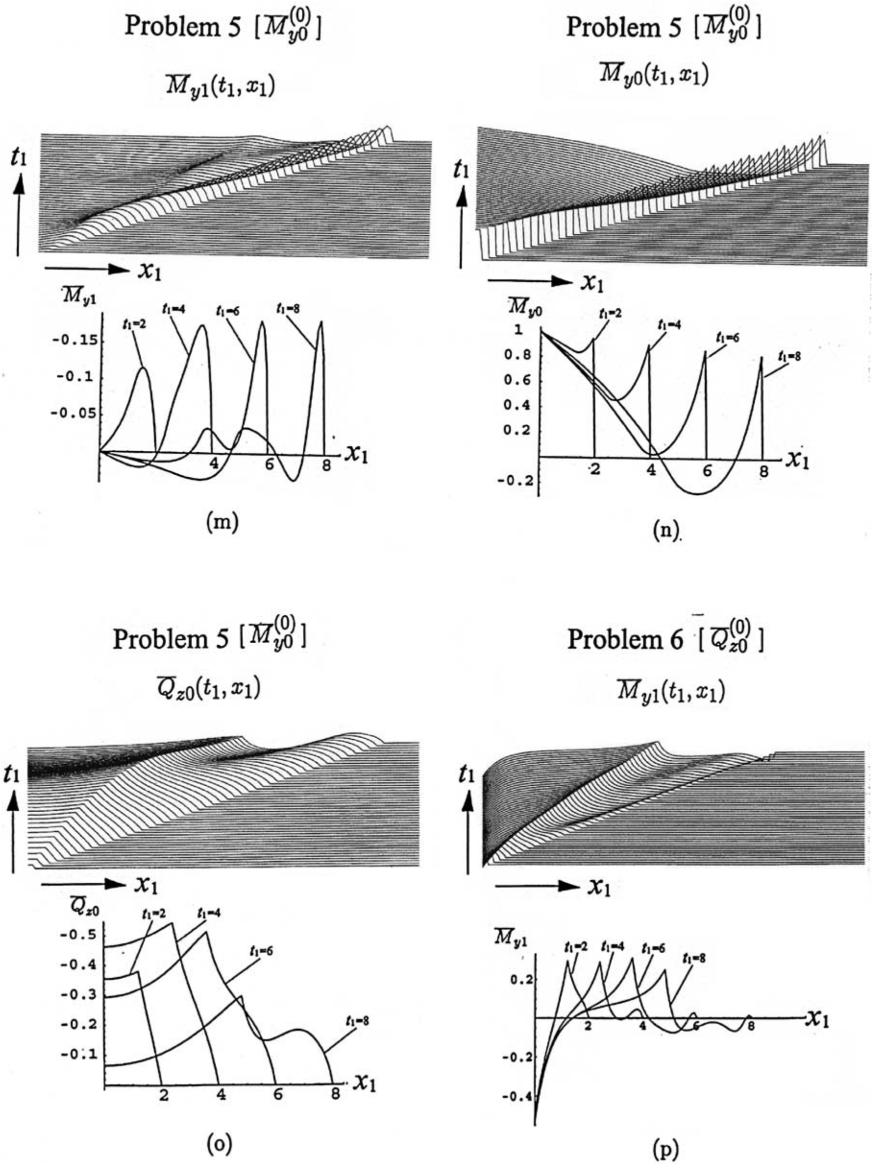

Fig. 12 (*continued*)

ordinary state quantities. Positive and negative are reinversed at the portion of the preceding longitudinal wave and the following transverse wave. From this, we know that the direction of nonlinear warping displacement is reversed. The usual bending moment and the new warping moment are made identical in their warping resistance moment; therefore, a simple comparison of their quantities can be made. In our theory, from Fig. 11 we can estimate the effect of nonlinear warping to be approximately 10–20% of the conventional linear warping.

Fig. 12 shows how the waves of stress resultants propagate with time. The left edge of our side is a free end of a semi-infinite beam to which a transverse impact is to be applied. The beam axis extends infinitely to

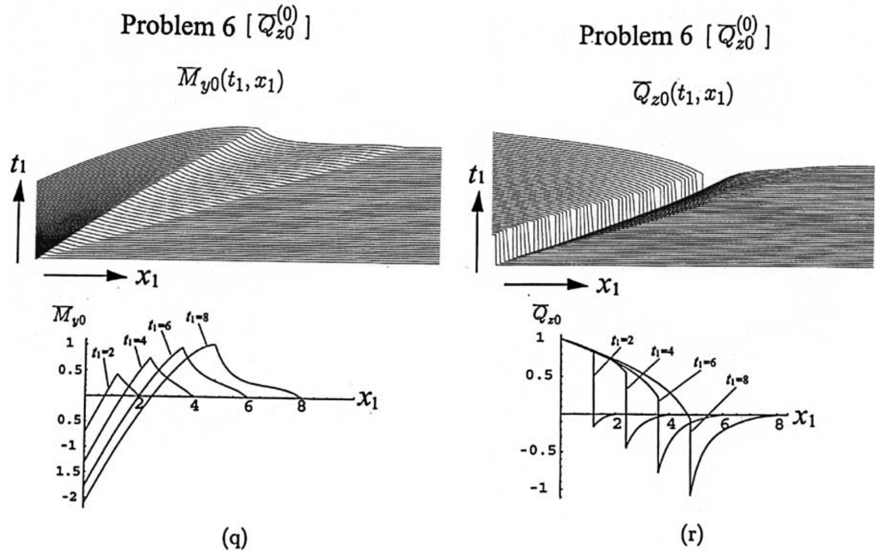

Fig. 12 (*continued*)

the right. With time, the wave propagation of each stress resultant toward the depth direction on the paper can be observed in each problem. The wave propagating with longitudinal velocity c_1 is quick and thus leads the other. The wave that propagates at the transversal velocity c_2 is slow and thus follows the leading wave. However, the crest of the transversal wave lags behind the leading edge of the longitudinal wave; thereby, the distance between the two eventually increases. The distribution of the stress resultants in the beam-axial direction is indicated at specific times $t_1 = 2$, 4, 6, 8 shown in the lower part of Fig. 12.

8. Conclusions

We have derived a higher-order beam theory using the Reissner functional and considering nonlinear warping, and, on the basis of its governing equation, analyzed the effect of nonlinear warping. Our solution uses the Laplace transform technique, which becomes an exact solution within the assumed range of the governing differential equation. In theory, any terms of higher-order nonlinear warping can be included. However, mathematically, calculations including a complex integral are very complicated and require enormous effort. In this study, we considered only the first term of nonlinear warping and solved the solution for space for which the matrices are Laplace-transformed as well as inversely transformed. As a numerical example, we dealt with suddon transverse loading to the edge of a semi-infinite beam.

Although the integrand of the complex integral is complicated, its value is controlled by the transition in value of several kinds of characteristic roots on the real and imaginary axes. Namely, so long as we clarify the passage of changes of characteristic roots into other characteristic roots along the branch cut, we can trace the transition of the integrand itself. By grasping these specific characteristics, we can obtain an analytical, correct, and rapid solution without use of numerical integration. For this paper, we used mathematical processing software, *Mathematica*, for the analytical calculation of integration. The following results were obtained:

(1) In the integrand which appears at the Laplace inverse transform for the impact problem of a beam, the branch point and pole are identical.

(2) Depending on the types of characteristic roots, specific section between some branch points on the real and imaginary axes may or may not contribute to the integral value.
(3) The pole of order 1/2 and the pole of order 1 that is affected by bifurcation at the origin do not contribute to the integral values; however, the pole of order 1 that is not affected by bifurcation at the origin does contribute to the integral value as a constant. The pole of order 3/2 contributes to the integral value as a linear function of x.

By applying a transverse impact to the edge of a semi-infinite beam, we investigated beam behavior. Then, we compared the solution of linear warping of the B/E beam, that of the Timoshenko beam, and that of our higher-order beam theory. The results are as follows:

(1) As compared with the B/E beam theory, the Timoshenko beam theory and the higher-order beam theory yield superior results.
(2) The solution of the higher-order beam which includes nonlinear warping shows a better result than that of the Timoshenko beam with only linear warping.
(3) Cross-sectional rotation and warping moment for nonlinear warping show similar distribution in the axial direction, except for the vicinity of the beam edge.
(4) As compared with the conventional state functions, the state functions for nonlinear warping are more radical in change between positive and negative; however, the degree of effect is about 10–20% that of linear warping.
(5) The components of nonlinear warping for transversal and longitudinal wave are, in general, opposite in direction.

In ordinary space, elements of the field transfer matrix exhibit symmetry with respect to the subsidiary diagonal line if stress resultants and deformations are properly defined. In this study, we have shown that, even in the wave-number domain where the governing equation is Laplace-transformed or in the time domain which is inversely transformed, symmetry with respect to the subsidiary diagonal line, the so-called *reciprocal relation*, can be obtained by properly defining the stress resultants. We have shown this analytically and with numerical examples.

In this paper, only flexural wave propagation along the beam axis accompanied by vertical deflection was treated. Coupling problems with other modes, i.e. longitudinal wave, torsional wave and flexural wave accompanied by horizontal deflection, can be solved by superposition after orthogonalization of the general eigenvalue problem. Treatment of other deformation modes is generally the same as in this paper, except for minor changes to the differential equations.

The formulation was treated rigorously in this paper. To the best knowledge of the authors, there have been very few closed-form solutions of transfer matrix elements from sixth-order differential equations. Such a solution and procedure is expected to provide many beneficial suggestions to near-field problems. This analytical solution can be used to examine the accuracy of approximate numerical solutions. Furthermore, as the transfer matrix elements are convertible to a stiffness matrix, closed-form solutions of the stiffness matrix elements can be readily obtained. Here, two solutions were given; matrix elements for a finite beam, and matrix elements for a semi-infinite beam. For extremely long beams, Usuki and Nakamura's *modified transfer matrix method* (Usuki and Nakamura, 1986) can be used.

Appendix A. Elements of field transfer matrix for a semi-infinite beam in frequency domain

$$t_{11} = t_{66} = \sum_{j=1}^{3} \frac{\left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{21} = t_{65} = \sum_{j=1}^{3} \frac{p^2\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{31} = t_{64} = -\frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2(\lambda_j^2 - p^2)}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{41} = t_{63} = -\frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2(\lambda_j^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{51} = t_{62} = \sum_{j=1}^{3} \frac{p^2\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{61} = -\sum_{j=1}^{3} \frac{p^2\left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{12} = t_{56} = -\sum_{j=1}^{3} \frac{\lambda_j\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{22} = t_{55} = -\sum_{j=1}^{3} \frac{-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{32} = t_{54} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{42} = t_{53} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{\lambda_j p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{52} = -\sum_{j=1}^{3} \frac{\lambda_j\left[-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)\right]}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{13} = t_{46} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{\lambda_j(\lambda_j^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{23} = t_{45} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$t_{33} = t_{44} = -\sum_{j=1}^{3} \frac{p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{43} = -\sum_{j=1}^{3} \frac{\lambda_j \left[p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)\right]}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{14} = t_{36} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{(\lambda_j^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{24} = t_{35} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2}{\lambda_j \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{34} = -\sum_{j=1}^{3} \frac{p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)}{\lambda_j \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{15} = t_{26} = -\sum_{j=1}^{3} \frac{\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{25} = -\sum_{j=1}^{3} \frac{-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)}{\lambda_j (\lambda_{j+2}^2 - \lambda_j^2)(\lambda_j^2 - \lambda_{j+1}^2)} \mathrm{e}^{-\lambda_j x_1},$$

$$t_{16} = -\sum_{j=1}^{3} \frac{\lambda_j \left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{p^2 \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1}.$$

Appendix B. Elements of field transfer matrix for a semi-infinite beam in time domain

$$\bar{t}_{11} = \bar{t}_{66} = \sum_{j=1}^{3} \frac{\left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{p \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{21} = \bar{t}_{65} = \sum_{j=1}^{3} \frac{p\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\lambda_j \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{31} = \bar{t}_{64} = -\frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p(\lambda_j^2 - p^2)}{\lambda_j \left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{41} = \bar{t}_{63} = -\frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{(\lambda_j^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{51} = \bar{t}_{62} = \sum_{j=1}^{3} \frac{\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{61} = -\sum_{j=1}^{3} \frac{\left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{12} = \bar{t}_{56} = -\sum_{j=1}^{3} \frac{\lambda_j\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{p\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{22} = \bar{t}_{55} = -\sum_{j=1}^{3} \frac{-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)}{p\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{32} = \bar{t}_{54} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{42} = \bar{t}_{53} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{\lambda_j}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{52} = -\sum_{j=1}^{3} \frac{\lambda_j\left[-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)\right]}{p^2\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{13} = \bar{t}_{46} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{\lambda_j(\lambda_j^2 - p^2)}{p\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{23} = \bar{t}_{45} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{33} = \bar{t}_{44} = -\sum_{j=1}^{3} \frac{p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)}{p\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{43} = -\sum_{j=1}^{3} \frac{\lambda_j\left[p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)\right]}{p^2\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} e^{-\lambda_j x_1},$$

$$\bar{t}_{14} = \bar{t}_{36} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{(\lambda_j^2 - p^2)}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{24} = \bar{t}_{35} = \frac{k_{10}}{k_{00}} \sum_{j=1}^{3} \frac{p^2}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{34} = -\sum_{j=1}^{3} \frac{p^2 + (\lambda_j^2 - p^2)(\lambda_j^2 - \gamma p^2)}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{15} = \bar{t}_{26} = -\sum_{j=1}^{3} \frac{\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2}{\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{25} = -\sum_{j=1}^{3} \frac{-p^2 + (\lambda_{j+1}^2 - p^2)(\lambda_{j+2}^2 - p^2)}{\lambda_j\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1},$$

$$\bar{t}_{16} = -\sum_{j=1}^{3} \frac{\lambda_j\left(\lambda_j^2 - p^2\right)\left[\lambda_{j+1}^2 + \lambda_{j+2}^2 - (1+\gamma)p^2\right]}{p^2\left(\lambda_{j+2}^2 - \lambda_j^2\right)\left(\lambda_j^2 - \lambda_{j+1}^2\right)} \mathrm{e}^{-\lambda_j x_1}.$$

Appendix C. Integral expressions used in Table 5

$$\begin{aligned}
I_{11}^{\mathrm{R}} = I_{66}^{\mathrm{R}} = {} & -2\int_0^{p_1} \left\{ \frac{(\lambda_1^2 - p^2)[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{p(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right. \\
& \left. - \frac{(\lambda_2^2 - p^2)[\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2]}{p(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho \\
& + 2\int_{p_2}^{p_3} \left\{ \frac{(\lambda_3^2 - p^2)[\lambda_1^2 + \lambda_2^2 - (1+\gamma)p^2]}{p(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh(\lambda_3 x_1 - \rho t_1) \right. \\
& \left. - \frac{(\lambda_1^2 - p^2)[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{p(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,
\end{aligned}$$

$$\begin{aligned}
I_{11}^{\mathrm{I}} = I_{66}^{\mathrm{I}} = {} & 4\int_0^{p_{i2}} \frac{(\lambda_2^2 - p^2)[\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2]}{p(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \sinh \lambda_2 x_1 \sin \rho t_1 \,\mathrm{d}\rho \\
& + 4\int_0^{p_{i3}} \frac{(\lambda_3^2 - p^2)[\lambda_1^2 + \lambda_2^2 - (1+\gamma)p^2]}{p(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \sinh \lambda_3 x_1 \sin \rho t_1 \,\mathrm{d}\rho,
\end{aligned}$$

$$I_{21}^{\mathrm{R}} = I_{65}^{\mathrm{R}} = 2\int_0^{p_1} \left\{ \frac{p[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) - \frac{p[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \sinh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$
$$- 2\int_{p_2}^{p_3} \left\{ \frac{p[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \sinh(\lambda_3 x_1 - \rho t_1) - \frac{p[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{21}^{\mathrm{I}} = I_{65}^{\mathrm{I}} = -4\int_0^{p_{i2}} \frac{p[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \cosh\lambda_2 x_1 \sin\rho t_1 \,\mathrm{d}\rho$$
$$- 4\int_0^{p_{i3}} \frac{p[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \cosh\lambda_3 x_1 \sin\rho t_1 \,\mathrm{d}\rho,$$

$$I_{31}^{\mathrm{R}} = I_{64}^{\mathrm{R}}$$
$$= -2\frac{k_{10}}{k_{00}}\int_0^{p_1} \left\{ \frac{p(\lambda_1^2-p^2)}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) - \frac{p(\lambda_2^2-p^2)}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \sinh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$
$$+ 2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3} \left\{ \frac{p(\lambda_3^2-p^2)}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \sinh(\lambda_3 x_1 - \rho t_1) - \frac{p(\lambda_1^2-p^2)}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{31}^{\mathrm{I}} = I_{64}^{\mathrm{I}} = 4\frac{k_{10}}{k_{00}}\int_0^{p_{i2}} \frac{p(\lambda_2^2-p^2)}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \cosh\lambda_2 x_1 \sin\rho t_1 \,\mathrm{d}\rho$$
$$+ 4\frac{k_{10}}{k_{00}}\int_0^{p_{i3}} \frac{p(\lambda_3^2-p^2)}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \cosh\lambda_3 x_1 \sin\rho t_1 \,\mathrm{d}\rho,$$

$$I_{41}^{\mathrm{R}} = I_{63}^{\mathrm{R}} = -I_{14}^{\mathrm{R}} = -I_{36}^{\mathrm{R}}$$
$$= -2\frac{k_{10}}{k_{00}}\int_0^{p_1} \left\{ \frac{(\lambda_1^2-p^2)}{(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) - \frac{(\lambda_2^2-p^2)}{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \sinh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$
$$+ 2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3} \left\{ \frac{(\lambda_3^2-p^2)}{(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \sinh(\lambda_3 x_1 - \rho t_1) - \frac{(\lambda_1^2-p^2)}{(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{41}^{\mathrm{I}} = I_{63}^{\mathrm{I}} = -I_{14}^{\mathrm{I}} = -I_{36}^{\mathrm{I}} = 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i2}} \frac{(\lambda_2^2-p^2)}{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \sinh\lambda_2 x_1 \cos\rho t_1 \,\mathrm{d}\rho$$
$$+ 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i3}} \frac{(\lambda_3^2-p^2)}{(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \sinh\lambda_3 x_1 \cos\rho t_1 \,\mathrm{d}\rho,$$

$$I_{51}^{\mathrm{R}} = I_{62}^{\mathrm{R}} = -I_{15}^{\mathrm{R}} = -I_{26}^{\mathrm{R}}$$
$$= 2\int_0^{p_1} \left\{ \frac{\lambda_2^2+\lambda_3^2-(1+\gamma)p^2}{(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) - \frac{\lambda_3^2+\lambda_1^2-(1+\gamma)p^2}{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)} \sinh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$
$$- 2\int_{p_2}^{p_3} \left\{ \frac{\lambda_1^2+\lambda_2^2-(1+\gamma)p^2}{(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)} \sinh(\lambda_3 x_1 - \rho t_1) - \frac{\lambda_2^2+\lambda_3^2-(1+\gamma)p^2}{(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)} \sinh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{51}^{\mathrm{I}} = I_{62}^{\mathrm{I}} = -I_{15}^{\mathrm{I}} = -I_{26}^{\mathrm{I}}$$
$$= -4\mathrm{i}\int_0^{p_{i2}} \frac{\lambda_3^2+\lambda_1^2-(1+\gamma)p^2}{(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\sinh\lambda_2 x_1\cos\rho t_1\,\mathrm{d}\rho - 4\mathrm{i}\int_0^{p_{i3}} \frac{\lambda_1^2+\lambda_2^2-(1+\gamma)p^2}{(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\sinh\lambda_3 x_1\cos\rho t_1\,\mathrm{d}\rho,$$

$$\begin{aligned}I_{61}^{\mathrm{R}} = {} & 2\int_0^{p_1}\left\{\frac{(\lambda_1^2-p^2)[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1x_1-\rho t_1)\right.\\ & \left. - \frac{(\lambda_2^2-p^2)[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh(\lambda_2x_1-\rho t_1)\right\}\mathrm{d}\rho\\ & - 2\int_{p_2}^{p_3}\left\{\frac{(\lambda_3^2-p^2)[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh(\lambda_3x_1-\rho t_1)\right.\\ & \left. - \frac{(\lambda_1^2-p^2)[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{\lambda_1(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1x_1-\rho t_1)\right\}\mathrm{d}\rho,\end{aligned}$$

$$\begin{aligned}I_{61}^{\mathrm{I}} = {} & -4\mathrm{i}\int_0^{p_{i2}}\frac{(\lambda_2^2-p^2)[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{\lambda_2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh\lambda_2x_1\cos\rho t_1\,\mathrm{d}\rho\\ & - 4\mathrm{i}\int_0^{p_{i3}}\frac{(\lambda_3^2-p^2)[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{\lambda_3(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh\lambda_3x_1\cos\rho t_1\,\mathrm{d}\rho,\end{aligned}$$

$$\begin{aligned}I_{12}^{\mathrm{R}} = {} & I_{56}^{\mathrm{R}}\\ = {} & -2\int_0^{p_1}\left\{\frac{\lambda_1[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\sinh(\lambda_1x_1-\rho t_1) - \frac{\lambda_2[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\sinh(\lambda_2x_1-\rho t_1)\right\}\mathrm{d}\rho\\ & + 2\int_{p_2}^{p_3}\left\{\frac{\lambda_3[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\sinh(\lambda_3x_1-\rho t_1) - \frac{\lambda_1[\lambda_2^2+\lambda_3^2-(1+\gamma)p^2]}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\sinh(\lambda_1x_1-\rho t_1)\right\}\mathrm{d}\rho,\end{aligned}$$

$$\begin{aligned}I_{12}^{\mathrm{I}} = I_{56}^{\mathrm{I}} = {} & 4\int_0^{p_{i2}}\frac{\lambda_2[\lambda_3^2+\lambda_1^2-(1+\gamma)p^2]}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh\lambda_2x_1\sin\rho t_1\,\mathrm{d}\rho\\ & + 4\int_0^{p_{i3}}\frac{\lambda_3[\lambda_1^2+\lambda_2^2-(1+\gamma)p^2]}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh\lambda_3x_1\sin\rho t_1\,\mathrm{d}\rho,\end{aligned}$$

$$\begin{aligned}I_{22}^{\mathrm{R}} = I_{55}^{\mathrm{R}} = {} & 2\int_0^{p_1}\left\{\frac{-p^2+(\lambda_2^2-p^2)(\lambda_3^2-p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1x_1-\rho t_1)\right.\\ & \left. - \frac{-p^2+(\lambda_3^2-p^2)(\lambda_1^2-p^2)}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh(\lambda_2x_1-\rho t_1)\right\}\mathrm{d}\rho\\ & - 2\int_{p_2}^{p_3}\left\{\frac{-p^2+(\lambda_1^2-p^2)(\lambda_2^2-p^2)}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh(\lambda_3x_1-\rho t_1)\right.\\ & \left. - \frac{-p^2+(\lambda_2^2-p^2)(\lambda_3^2-p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1x_1-\rho t_1)\right\}\mathrm{d}\rho,\end{aligned}$$

$$I^{\mathrm{I}}_{22} = I^{\mathrm{I}}_{55} = -4\int_0^{p_{i2}} \frac{-p^2 + (\lambda_3^2 - p^2)(\lambda_1^2 - p^2)}{p(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \sinh\lambda_2 x_1 \sin\rho t_1 \,\mathrm{d}\rho$$
$$- 4\int_0^{p_{i3}} \frac{-p^2 + (\lambda_1^2 - p^2)(\lambda_2^2 - p^2)}{p(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \sinh\lambda_3 x_1 \sin\rho t_1 \,\mathrm{d}\rho,$$

$$I^{\mathrm{R}}_{32} = I^{\mathrm{R}}_{23} = I^{\mathrm{R}}_{45} = I^{\mathrm{R}}_{54}$$
$$= -2\frac{k_{10}}{k_{00}}\int_0^{p_1}\left\{\frac{p}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1) - \frac{p}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}\cosh(\lambda_2 x_1 - \rho t_1)\right\}\mathrm{d}\rho$$
$$+ 2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3}\left\{\frac{p}{(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)}\cosh(\lambda_3 x_1 - \rho t_1) - \frac{p}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1)\right\}\mathrm{d}\rho,$$

$$I^{\mathrm{I}}_{32} = I^{\mathrm{I}}_{23} = I^{\mathrm{I}}_{45} = I^{\mathrm{I}}_{54} = 4\frac{k_{10}}{k_{00}}\int_0^{p_{i2}} \frac{p}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \sinh\lambda_2 x_1 \sin\rho t_1 \,\mathrm{d}\rho$$
$$+ 4\frac{k_{10}}{k_{00}}\int_0^{p_{i3}} \frac{p}{(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \sinh\lambda_3 x_1 \sin\rho t_1 \,\mathrm{d}\rho,$$

$$I^{\mathrm{R}}_{42} = I^{\mathrm{R}}_{53}$$
$$= -2\frac{k_{10}}{k_{00}}\int_0^{p_1}\left\{\frac{\lambda_1}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1) - \frac{\lambda_2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}\cosh(\lambda_2 x_1 - \rho t_1)\right\}\mathrm{d}\rho$$
$$+ 2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3}\left\{\frac{\lambda_3}{(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)}\cosh(\lambda_3 x_1 - \rho t_1) - \frac{\lambda_1}{(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1)\right\}\mathrm{d}\rho,$$

$$I^{\mathrm{I}}_{42} = I^{\mathrm{I}}_{53} = 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i2}} \frac{\lambda_2}{(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh\lambda_2 x_1 \cos\rho t_1 \,\mathrm{d}\rho$$
$$+ 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i3}} \frac{\lambda_3}{(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh\lambda_3 x_1 \cos\rho t_1 \,\mathrm{d}\rho,$$

$$I^{\mathrm{R}}_{52} = 2\int_0^{p_1}\left\{\frac{\lambda_1[-p^2 + (\lambda_2^2 - p^2)(\lambda_3^2 - p^2)]}{p^2(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1)\right.$$
$$\left. - \frac{\lambda_2[-p^2 + (\lambda_3^2 - p^2)(\lambda_1^2 - p^2)]}{p^2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)}\cosh(\lambda_2 x_1 - \rho t_1)\right\}\mathrm{d}\rho$$
$$- 2\int_{p_2}^{p_3}\left\{\frac{\lambda_3[-p^2 + (\lambda_1^2 - p^2)(\lambda_2^2 - p^2)]}{p^2(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)}\cosh(\lambda_3 x_1 - \rho t_1)\right.$$
$$\left. - \frac{\lambda_1[-p^2 + (\lambda_2^2 - p^2)(\lambda_3^2 - p^2)]}{p^2(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)}\cosh(\lambda_1 x_1 - \rho t_1)\right\}\mathrm{d}\rho,$$

$$I_{52}^{\mathrm{I}} = -4\mathrm{i}\int_0^{p_{i2}} \frac{\lambda_2[-p^2+(\lambda_3^2-p^2)(\lambda_1^2-p^2)]}{p^2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh\lambda_2 x_1 \cos\rho t_1\,\mathrm{d}\rho$$
$$-4\mathrm{i}\int_0^{p_{i3}} \frac{\lambda_3[-p^2+(\lambda_1^2-p^2)(\lambda_2^2-p^2)]}{p^2(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh\lambda_3 x_1 \cos\rho t_1\,\mathrm{d}\rho,$$

$$I_{13}^{\mathrm{R}} = I_{46}^{\mathrm{R}}$$
$$= 2\frac{k_{10}}{k_{00}}\int_0^{p_1}\left\{\frac{\lambda_1(\lambda_1^2-p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\sinh(\lambda_1 x_1-\rho t_1) - \frac{\lambda_2(\lambda_2^2-p^2)}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\sinh(\lambda_2 x_1-\rho t_1)\right\}\mathrm{d}\rho$$
$$-2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3}\left\{\frac{\lambda_3(\lambda_3^2-p^2)}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\sinh(\lambda_3 x_1-\rho t_1) - \frac{\lambda_1(\lambda_1^2-p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\sinh(\lambda_1 x_1-\rho t_1)\right\}\mathrm{d}\rho,$$

$$I_{13}^{\mathrm{I}} = I_{46}^{\mathrm{I}} = -4\frac{k_{10}}{k_{00}}\int_0^{p_{i2}}\frac{\lambda_2(\lambda_2^2-p^2)}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh\lambda_2 x_1\sin\rho t_1\,\mathrm{d}\rho$$
$$-4\frac{k_{10}}{k_{00}}\int_0^{p_{i3}}\frac{\lambda_3(\lambda_3^2-p^2)}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh\lambda_3 x_1\sin\rho t_1\,\mathrm{d}\rho,$$

$$I_{33}^{\mathrm{R}} = I_{44}^{\mathrm{R}}$$
$$= 2\int_0^{p_1}\left\{\frac{p^2+(\lambda_1^2-p^2)(\lambda_1^2-\gamma p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1 x_1-\rho t_1) - \frac{p^2+(\lambda_2^2-p^2)(\lambda_2^2-\gamma p^2)}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh(\lambda_2 x_1-\rho t_1)\right\}\mathrm{d}\rho$$
$$-2\int_{p_2}^{p_3}\left\{\frac{p^2+(\lambda_3^2-p^2)(\lambda_3^2-\gamma p^2)}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh(\lambda_3 x_1-\rho t_1) - \frac{p^2+(\lambda_1^2-p^2)(\lambda_1^2-\gamma p^2)}{p(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1 x_1-\rho t_1)\right\}\mathrm{d}\rho,$$

$$I_{33}^{\mathrm{I}} = I_{44}^{\mathrm{I}} = -4\int_0^{p_{i2}}\frac{p^2+(\lambda_2^2-p^2)(\lambda_2^2-\gamma p^2)}{p(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\sinh\lambda_2 x_1\sin\rho t_1\,\mathrm{d}\rho$$
$$-4\int_0^{p_{i3}}\frac{p^2+(\lambda_3^2-p^2)(\lambda_3^2-\gamma p^2)}{p(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\sinh\lambda_3 x_1\sin\rho t_1\,\mathrm{d}\rho,$$

$$I_{43}^{\mathrm{R}} = 2\int_0^{p_1}\left\{\frac{\lambda_1[p^2+(\lambda_1^2-p^2)(\lambda_1^2-\gamma p^2)]}{p^2(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1 x_1-\rho t_1)\right.$$
$$\left.-\frac{\lambda_2[p^2+(\lambda_2^2-p^2)(\lambda_2^2-\gamma p^2)]}{p^2(\lambda_1^2-\lambda_2^2)(\lambda_2^2-\lambda_3^2)}\cosh(\lambda_2 x_1-\rho t_1)\right\}\mathrm{d}\rho$$
$$-2\int_{p_2}^{p_3}\left\{\frac{\lambda_3[p^2+(\lambda_3^2-p^2)(\lambda_3^2-\gamma p^2)]}{p^2(\lambda_2^2-\lambda_3^2)(\lambda_3^2-\lambda_1^2)}\cosh(\lambda_3 x_1-\rho t_1)\right.$$
$$\left.-\frac{\lambda_1[p^2+(\lambda_1^2-p^2)(\lambda_1^2-\gamma p^2)]}{p^2(\lambda_3^2-\lambda_1^2)(\lambda_1^2-\lambda_2^2)}\cosh(\lambda_1 x_1-\rho t_1)\right\}\mathrm{d}\rho,$$

$$I_{43}^{\mathrm{I}} = -4\mathrm{i}\int_0^{p_{i2}} \frac{\lambda_2[p^2 + (\lambda_2^2 - p^2)(\lambda_2^2 - \gamma p^2)]}{p^2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh \lambda_2 x_1 \cos \rho t_1 \, \mathrm{d}\rho$$

$$- 4\mathrm{i}\int_0^{p_{i3}} \frac{\lambda_3[p^2 + (\lambda_3^2 - p^2)(\lambda_3^2 - \gamma p^2)]}{p^2(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh \lambda_3 x_1 \cos \rho t_1 \, \mathrm{d}\rho,$$

$$I_{24}^{\mathrm{R}} = I_{35}^{\mathrm{R}}$$

$$= -2\frac{k_{10}}{k_{00}}\int_0^{p_1} \left\{ \frac{p^2}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) - \frac{p^2}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$

$$+ 2\frac{k_{10}}{k_{00}}\int_{p_2}^{p_3} \left\{ \frac{p^2}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh(\lambda_3 x_1 - \rho t_1) - \frac{p^2}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{24}^{\mathrm{I}} = I_{35}^{\mathrm{I}} = 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i2}} \frac{p^2}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh \lambda_2 x_1 \cos \rho t_1 \, \mathrm{d}\rho$$

$$+ 4\mathrm{i}\frac{k_{10}}{k_{00}}\int_0^{p_{i3}} \frac{p^2}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh \lambda_3 x_1 \cos \rho t_1 \, \mathrm{d}\rho,$$

$$I_{34}^{\mathrm{R}} = 2\int_0^{p_1} \left\{ \frac{p^2 + (\lambda_1^2 - p^2)(\lambda_1^2 - \gamma p^2)}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) - \frac{p^2 + (\lambda_2^2 - p^2)(\lambda_2^2 - \gamma p^2)}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$

$$- 2\int_{p_2}^{p_3} \left\{ \frac{p^2 + (\lambda_3^2 - p^2)(\lambda_3^2 - \gamma p^2)}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh(\lambda_3 x_1 - \rho t_1) - \frac{p^2 + (\lambda_1^2 - p^2)(\lambda_1^2 - \gamma p^2)}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{34}^{\mathrm{I}} = -4\mathrm{i}\int_0^{p_{i2}} \frac{p^2 + (\lambda_2^2 - p^2)(\lambda_2^2 - \gamma p^2)}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh \lambda_2 x_1 \cos \rho t_1 \, \mathrm{d}\rho$$

$$- 4\mathrm{i}\int_0^{p_{i3}} \frac{p^2 + (\lambda_3^2 - p^2)(\lambda_3^2 - \gamma p^2)}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh \lambda_3 x_1 \cos \rho t_1 \, \mathrm{d}\rho,$$

$$I_{25}^{\mathrm{R}} = 2\int_0^{p_1} \left\{ \frac{-p^2 + (\lambda_2^2 - p^2)(\lambda_3^2 - p^2)}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) - \frac{-p^2 + (\lambda_3^2 - p^2)(\lambda_1^2 - p^2)}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho$$

$$- 2\int_{p_2}^{p_3} \left\{ \frac{-p^2 + (\lambda_1^2 - p^2)(\lambda_2^2 - p^2)}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh(\lambda_3 x_1 - \rho t_1) - \frac{-p^2 + (\lambda_2^2 - p^2)(\lambda_3^2 - p^2)}{\lambda_1(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{25}^{\mathrm{I}} = -4\mathrm{i}\int_0^{p_{i2}} \frac{-p^2 + (\lambda_3^2 - p^2)(\lambda_1^2 - p^2)}{\lambda_2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh \lambda_2 x_1 \cos \rho t_1 \, \mathrm{d}\rho$$

$$- 4\mathrm{i}\int_0^{p_{i3}} \frac{-p^2 + (\lambda_1^2 - p^2)(\lambda_2^2 - p^2)}{\lambda_3(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh \lambda_3 x_1 \cos \rho t_1 \, \mathrm{d}\rho,$$

$$I_{16}^{\mathrm{R}} = 2\int_0^{p_1} \left\{ \frac{\lambda_1(\lambda_1^2 - p^2)[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{p^2(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) - \frac{\lambda_2(\lambda_2^2 - p^2)[\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2]}{p^2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh(\lambda_2 x_1 - \rho t_1) \right\} \mathrm{d}\rho - 2\int_{p_2}^{p_3} \left\{ \frac{\lambda_3(\lambda_3^2 - p^2)[\lambda_1^2 + \lambda_2^2 - (1+\gamma)p^2]}{p^2(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh(\lambda_3 x_1 - \rho t_1) - \frac{\lambda_1(\lambda_1^2 - p^2)[\lambda_2^2 + \lambda_3^2 - (1+\gamma)p^2]}{p^2(\lambda_3^2 - \lambda_1^2)(\lambda_1^2 - \lambda_2^2)} \cosh(\lambda_1 x_1 - \rho t_1) \right\} \mathrm{d}\rho,$$

$$I_{16}^{\mathrm{I}} = -4\mathrm{i}\int_0^{p_{i2}} \frac{\lambda_2(\lambda_2^2 - p^2)[\lambda_3^2 + \lambda_1^2 - (1+\gamma)p^2]}{p^2(\lambda_1^2 - \lambda_2^2)(\lambda_2^2 - \lambda_3^2)} \cosh\lambda_2 x_1 \cos\rho t_1 \,\mathrm{d}\rho - 4\mathrm{i}\int_0^{p_{i3}} \frac{\lambda_3(\lambda_3^2 - p^2)[\lambda_1^2 + \lambda_2^2 - (1+\gamma)p^2]}{p^2(\lambda_2^2 - \lambda_3^2)(\lambda_3^2 - \lambda_1^2)} \cosh\lambda_3 x_1 \cos\rho t_1 \,\mathrm{d}\rho.$$

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